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MARVELOUS SLICES OF ORTHOGONAL MATRICES

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ABSTRACT. The space of 4×4 special orthogonal matrices with zeros on the diagonal decomposes into the union of 14 irreducible surfaces whose intersections are beautifully encoded by the cuboctahedron. Using this decomposition, we exhibit a totally real witness set for SO(4). We explain how to obtain a similar decomposition for SO(5), where the 64 components can be grouped to obtain such a correspondence with the face lattice of a 3-polytope. We show that no such pattern exists for SO(6).

1. MAIN RESULT

In this note, we observe the beautiful geometry of 4×4 *hollow* special orthogonal matrices

$$\mathrm{HSO}(4) = \{M \in \mathrm{Mat}_{4\times4}(\mathbb{R}) \mid \mathrm{diag}(M) = (0,0,0,0) \text{ and } M \in \mathrm{SO}(4)\} \subseteq \mathbb{R}^{4\times4} = \mathbb{R}^{16}.$$

Our point of departure is a full description of the components of HSO(4) and their intersections.

Theorem 1.1. The variety $\mathrm{HSO}(4) \subseteq \mathbb{R}^{16}$ is the union of 14 irreducible surfaces: six tori of degree four and eight spheres of degree two. There exists a bijection between these components and the facets of the cuboctahedron $\mathcal{C}$ (illustrated on the left of Figure 1) satisfying

(1) The six tori $\mathrm{SO}(2) \times \mathrm{SO}(2) \cong \mathbb{S}^1 \times \mathbb{S}^1$ correspond to the six quadrilateral faces of $\mathcal{C}$.
(2) The eight spheres $\mathbb{S}^2$ correspond to the eight triangular faces of $\mathcal{C}$.
(3) Components intersect in a curve if and only if the corresponding facets share an edge.
(4) Components intersect in points if and only if the corresponding facets share a vertex.
(5) Curve intersections of components are copies of $\mathrm{SO}(2) \cong \mathbb{S}^1$.
(6) Point intersections are signed permutation matrices and their negatives, i.e. $\mathbb{S}^0$.

The relevant bijection is illustrated in Figure 2.

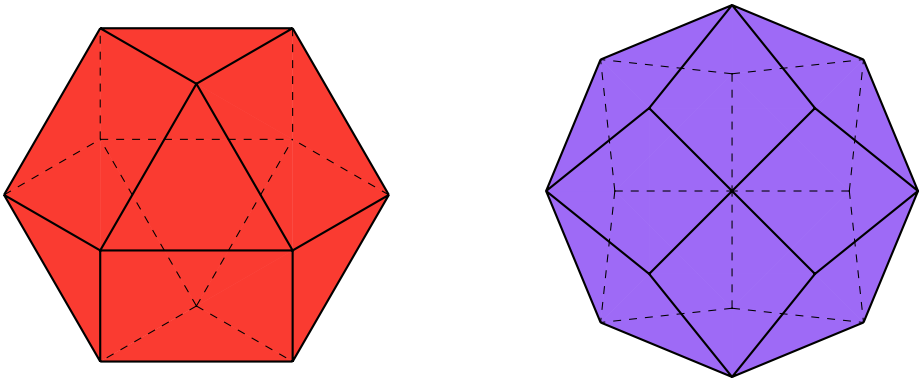


FIGURE 1. The cuboctahedron $\mathcal{C}$ (left) and another polytope $\mathcal{P}$ (right). Their facets are in bijection with components of HSO(4) and quadruples of components of $\mathrm{SO}^\star(5)$ respectively. The face lattice of each gives the intersection lattice of the corresponding components in agreement with Theorem 1.1 and Theorem 5.1.

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For SO(5), the story is similar. There is a zero pattern which slices SO(5) down to a union $\mathrm{SO}^\star(5)$ of surfaces. As with HSO(4), this slice is *degree-generic* in the sense that the sum of the degrees of the 64 components equals the degree 384 of SO(5) [BBB+17]. We put quadruples of components of $\mathrm{SO}^\star(5)$ in bijection with the facets of the polytope on the right of Figure 1 in such a way that an analogue of Theorem 1.1 holds. The full statement is given in Theorem 5.1.

In Section 2 we describe the 14 surfaces of HSO(4) and establish the bijection between these surfaces and the facets of the cuboctahedron $\mathcal{C}$. In Section 3.1 and Section 3.2, respectively, we describe the one-dimensional and zero-dimensional intersections of these components. In Section 4 we use this decomposition to find a totally real witness set for SO(4), resolving the conjecture [BBB+17, Conjecture 7.1] for $n = 4$. We extend this story to $n = 5$ in Section 5: we identify 64 components of $\mathrm{SO}^\star(5)$ which are in bijection with the 16 facets of the polytope $\mathcal{P}$ shown in the right of Figure 1 when grouped by fours. Just as in the case of $n = 4$, the face lattice of this polytope encodes the intersections between groups of components, as detailed in Theorem 5.1. We show the pattern does not continue for $n = 6$, see Theorem 6.2. All results in this paper, other than Theorem 6.2, can be verified by hand. For convenience, we provide code to verify all results using computer algebra software [GS26, OSC24, DEF+24, GJ00, MP13], available at [BG26].

1.1. **Related work and motivation.** Hollow matrices appear in several areas: we refer to [Gen07] for some examples. This includes the case of adjacency matrices of graphs, and that of dissimilarity matrices in statistics [Tro02]. Hollow orthogonal matrices arise when looking for normal forms of non-definite operators with desirable stability properties [Nic25] and in other combinatorial frameworks for which we refer to [Col10].

Our investigation began at the 2018 ICERM semester on nonlinear algebra. We sought to generalize the formula for the degree of the special orthogonal group in [BBB+17] to Stiefel manifolds. One strategy we considered was to compute the degrees of these homogeneous spaces by predicting their intersection with select linear spaces. The goal was to re-establish the SO(n) degree formula via enumerative combinatorics, rather than an application of Kazarnovskii's theorem [Kaz87]. This proved incredibly successful for SO(4) (as described in Theorem 1.1) since the hollow linear space decomposes SO(4) combinatorially. This program is successful for SO(5) as well (as described in Section 5). This idea, however, does not succeed to provide a combinatorial interpretation of the degree of SO(6) (see Theorem 6.2). Ultimately, the goal of determining the degrees of Stiefel manifolds was achieved using alternative techniques [BG21], leaving the *marvelous* phenomenon witnessed during this process to be shared in the present work.

Theorem 1.1 is used to provide a positive answer to the next unknown instance of a conjecture [BBB+17, Conjecture 7.1] concerning the real algebraic geometry of linear algebraic groups: does there exist a zero-dimensional linear slice of SO(n) consisting entirely of real points? The answer is 'yes' and the $\deg(\mathrm{SO}(4)) = 40$ real points are given explicitly in Section 4.

Our main motivations for this manuscript are as follows. First and foremost, we wanted to share these examples of HSO(4), and its generalization $\mathrm{SO}^\star(5)$ with the mathematical community; we could not find them in the literature and they pertain to mathematical objects of broad general interest. Secondly, we hope that our work will encourage investigation of the intrinsic geometric reason for why such decomposable slices exist for SO(4) and SO(5) and whether the phenomenon persists in some form for $n > 5$. Finally, we hope this serves as a motivating example to real algebraic geometers, showcasing the heuristic that structured combinatorial decompositions can make real algebro-geometric searches feasible.

Acknowledgements. We are thankful to Michael Joswig for his helpful input regarding the polytope $\mathcal{P}$. The initial discoveries in this paper were obtained during the 2018 ICERM semester on *Nonlinear Algebra* (NSF DMS-1439786). TB is currently supported by an NSERC Discovery Grant (RGPIN-2023-03551).

2. The fourteen surfaces of hollow SO(4)

We describe the components of HSO(4) and give the bijection with the facets of the cuboctahedron.

2.1. **Eight Spheres.** We define eight spheres in $\mathbb{R}^{16}$, all of which are parametrized by points (x, y, z) on the standard unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. One sphere, denoted S^+_{+++}, is the image of the map

$$S^+_{+++} \qquad \mathbb{S}^2 \ni (x,y,z) \mapsto \begin{bmatrix} 0 & x & -y & z \\ x & 0 & z & y \\ y & z & 0 & -x \\ z & -y & -x & 0 \end{bmatrix}. \tag{2.1}$$

This parametrization is summarized by the diagram



where $(\blacksquare, \blacksquare, \blacksquare) \in \mathbb{S}^2$, $\blacksquare = 0$, $(\blacksquare, \blacksquare, \blacksquare) = -(\blacksquare, \blacksquare, \blacksquare)$.

The variety S^+_{+++} is an irreducible surface of degree 2 and by inspection, $S^+_{+++} \subseteq \mathrm{HSO}(4)$.

There are seven other components, each indexed by an even 4-signature:

$$S^-_{-++}, S^-_{+-+}, S^-_{++-}, S^+_{--+}, S^+_{-+-}, S^+_{+--}, S^-_{---}.$$

The lower three signs correspond to negation of columns 2, 3, and 4 of (2.1), respectively, whereas the top sign indicates a negation of the first row. This top sign may be viewed as notational redundancy since its value must be chosen to make the determinant one instead of negative one. For example, the component S^-_{++-} is obtained via S^+_{+++} by negating the fourth column and first row:

$$S^-_{++-} \qquad \mathbb{S}^2 \ni (x,y,z) \mapsto \begin{bmatrix} 0 & -x & y & z \\ x & 0 & z & -y \\ y & z & 0 & x \\ z & -y & -x & 0 \end{bmatrix} \qquad \text{}.$$

2.2. **Six Tori.** There are three ways to partition a hollow 4×4 matrix into two 2×2 blocks:

$$T^{(12)(34)} = \begin{bmatrix} 0 & 0 & \Delta & \Delta \\ 0 & 0 & \Delta & \Delta \\ \square & \square & 0 & 0 \\ \square & \square & 0 & 0 \end{bmatrix}, \quad T^{(13)(24)} = \begin{bmatrix} 0 & \square & 0 & \square \\ \Delta & 0 & \Delta & 0 \\ 0 & \square & 0 & \square \\ \Delta & 0 & \Delta & 0 \end{bmatrix}, \quad T^{(14)(23)} = \begin{bmatrix} 0 & \square & \square & 0 \\ \Delta & 0 & 0 & \Delta \\ \Delta & 0 & 0 & \Delta \\ 0 & \square & \square & 0 \end{bmatrix}. \tag{2.2}$$

Each 2×2 submatrix marked by either Δ or $\square$ can hold a copy of an orthogonal 2×2 matrix of common determinant ± 1. This gives explicit parametrizations for six tori,

$$T^{(12)(34)}_+, T^{(12)(34)}_-, T^{(13)(24)}_+, T^{(13)(24)}_-, T^{(14)(23)}_+, T^{(14)(23)}_-,$$

via a copy of $\mathrm{SO}(2) \times \mathrm{SO}(2) \cong \mathbb{S}^1 \times \mathbb{S}^1$. For example $T^{(12)(34)}_+$ is parametrized as

$$T^{(12)(34)}_+ \qquad \mathbb{S}^1 \times \mathbb{S}^1 \ni ((x,y),(z,w)) \mapsto \begin{bmatrix} 0 & 0 & z & -w \\ 0 & 0 & w & z \\ x & -y & 0 & 0 \\ y & x & 0 & 0 \end{bmatrix}$$

and is summarized diagrammatically as



where $(\blacksquare, \blacksquare), (\blacksquare, \blacksquare) \in \mathbb{S}^1$, $\blacksquare = 0$, $(\blacksquare, \blacksquare, \blacksquare, \blacksquare) = -(\blacksquare, \blacksquare, \blacksquare, \blacksquare)$.

The other five tori have similar diagrammatic representations. Each torus has degree four. Since $\deg(\mathrm{SO}(4)) = 40$ [BBB+17] we have that

$$\deg(\mathrm{SO}(4)) = 40 = 8 \cdot 2 + 6 \cdot 4 = 8 \cdot \left(\deg(S^{\pm}_{\pm\pm\pm})\right) + 6 \cdot \left(\deg\left(T^{(ab)(cd))}_{\pm}\right)\right).$$

The orthogonal group SO(4) is a smooth variety. Therefore, linear sections with the expected codimension cannot have degree higher than $\deg \mathrm{SO}(4)$; see, e.g., [EH16, Corollary 2.5]. This guarantees that the 14 surfaces we have exhibited comprise all components of HSO(4).

Now that all components of HSO(4) have been identified, we give the bijection of Theorem 1.1 between the 14 facets of the cuboctahedron $\mathcal{C}$ and these 14 components pictorially in Figure 2.

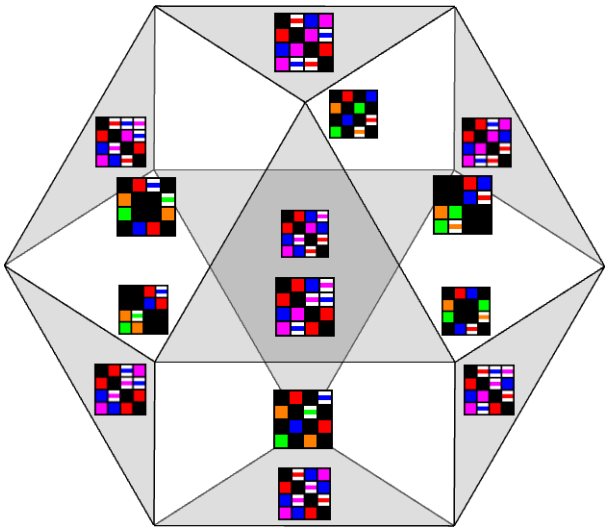


FIGURE 2. The cuboctahedron with facets labeled by the diagrams identifying components of HSO(4). This is the relevant bijection for Theorem 1.1.

3. Twenty-four curves and twelve pairs of points

3.1. **Twenty-four circles.** The 14 surfaces of HSO(4) intersect in dimension one according to Table 3.1. In particular, the tori never intersect each other in dimension one, nor do the spheres.

	$T_+^{(12)(34)}$	$T_-^{(12)(34)}$	$T_+^{(13)(24)}$	$T_-^{(13)(24)}$	$T_+^{(14)(23)}$	$T_-^{(14)(23)}$
S_{+++}^+		×		×		×
S_{-++}^-	×			×		×
S_{+-+}^-		×	×			×
S_{++-}^-		×		×	×	
S_{--+}^+	×		×			×
S_{-+-}^+	×			×	×	
S_{+--}^+		×	×		×	
S_{---}^-	×		×		×	

TABLE 3.1. The incidence matrix of surface components of HSO(4) that intersect in a curve.

Since the action of even column/row negation acts transitively on the spheres, we focus on analyzing the first row of the table, corresponding to S_{+++}^+. Imposing the three $2+2$ block patterns of (2.2) on the matrix (2.1) enforces x, y, or z (respectively) equal to zero, resulting in the matrix formats

$$T^{(12)(34)}: \begin{bmatrix} 0 & 0 & -y & z \\ 0 & 0 & z & y \\ y & z & 0 & 0 \\ z & -y & 0 & 0 \end{bmatrix}, \qquad T^{(13)(24)}: \begin{bmatrix} 0 & x & 0 & z \\ x & 0 & z & 0 \\ 0 & z & 0 & -x \\ z & 0 & -x & 0 \end{bmatrix}, \qquad T^{(14)(23)}: \begin{bmatrix} 0 & x & -y & 0 \\ x & 0 & 0 & y \\ y & 0 & 0 & -x \\ 0 & -y & -x & 0 \end{bmatrix}. \tag{3.1}$$

Each parametrization in (3.1) gives the diagonal embedding $\mathbb{S}^1 \hookrightarrow \mathbb{S}^1 \times \mathbb{S}^1 \cong T_\pm^{(ab)(cd)}$. The determinants of each 2×2 block of each of these circles are observed to be -1, verifying Table 3.1.

The incidence described in Table 3.1 of sphere-torus intersections of dimension one coincides with the incidence of triangle-quadrilateral intersections along edges of the cuboctahedron as in Figure 2. In particular, the 14 surface components of HSO(4) intersect in dimension one along 24 circles.

3.2. **Twelve pairs of fixed point free signed permutation matrices.** Having treated the two-dimensional components of HSO(4) and their one-dimensional intersections, we now turn toward their zero-dimensional intersections. As one would hope, these are encoded in the facet-vertex incidence of the cuboctahedron. The analysis of the previous section shows that sphere-torus pairs which do not intersect in curves, do not intersect at all. Thus, the only remaining possible non-empty intersections of components are between sphere-sphere pairs or torus-torus pairs.

The possible zero patterns occurring by intersecting the torus block patterns (2.2) are

$$\begin{bmatrix} 0 & \star & 0 & 0 \\ \star & 0 & 0 & 0 \\ 0 & 0 & 0 & \star \\ 0 & 0 & \star & 0 \end{bmatrix}$$

$$T^{(14)(23)} \qquad\qquad T^{(13)(24)}$$

$$\begin{bmatrix} 0 & 0 & \star & 0 \\ 0 & 0 & 0 & \star \\ \star & 0 & 0 & 0 \\ 0 & \star & 0 & 0 \end{bmatrix} \quad T^{(12)(34)} \quad \begin{bmatrix} 0 & 0 & 0 & \star \\ 0 & 0 & \star & 0 \\ 0 & \star & 0 & 0 \\ \star & 0 & 0 & 0 \end{bmatrix}$$

Matrices of these patterns which appear in HSO(4) are those which have the $\star$ symbols replaced with an even number of 1's and -1's. They may be thought of as copies of $\mathbb{S}^0 \cong \mathrm{O}(1)$ parametrized via the illustration in Figure 3. There are $3 \cdot 8 \cdot \frac{1}{2} = 12$ of them. These comprise the non-empty sphere-sphere intersections as well, and they appear as intersections in agreement with Figure 2. This fact completes the proof of Theorem 1.1.

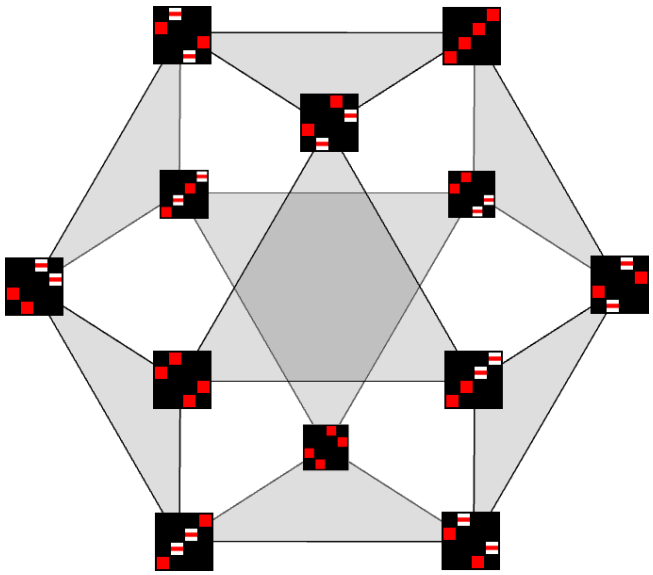


FIGURE 3. The vertices of $\mathcal{C}$ correspond to zero-dimensional intersections of components of HSO(4). Each represents a signed double transposition and its negative.

Each zero-dimensional intersection is a signed permutation matrix, paired with its negative. Since the matrix must be hollow, these permutations have no fixed points; however, not all fixed point free permutations arise. Only the double transpositions show up, whereas the six 4-cycles do not.

4. A totally real witness set

The decomposition of HSO(4) is relevant to real numerical algebraic geometry [SVW05] and semidefinite programming. In numerical algebraic geometry, varieties are represented on a computer as *witness sets*: zero-dimensional intersections with generic linear spaces of complementary dimension. If all such intersection points are real, we say the witness set is *totally real*. An open problem [BBB+17, Conjecture 7.1] is to determine if $\mathrm{SO}(n)$ admits a totally real witness set for all n. This conjecture relates to a low-rank semidefinite programming optimization problem, as described in [BBB+17, NRS10]. For an irreducible variety $X \subseteq \mathbb{C}^n$ and generic hyperplane $\mathcal{H}$, Bertini-like theorems (e.g. [Jou83, Theorem 6.3], [Har92, Section 11]) tell us what to expect of $X \cap \mathcal{H}$:

(1) (Degree) The degree of $\mathcal{H} \cap X$ is the degree of X.
(2) (Dimension) The dimension of $\mathcal{H} \cap X$ is $\dim(X) - 1$.
(3) (Irreducible) If $\dim(X) > 1$ then $\mathcal{H} \cap X$ is irreducible.

We refer to a hyperplane section as *degree-generic*, *dimension-generic*, and *irreducible-generic* accordingly. These definitions extend to linear sections of higher codimension. In the context of numerical algebraic geometry and witness sets, we require that $X \cap \mathcal{L}$ is obtained via a sequence of hyperplane sections which are both degree-generic and dimension-generic at each step. The previous section shows that the hollow hyperplanes $\{x_{i,i} = 0\}_{i=1}^4$ work as the first four slices for SO(4). Note that irreducible-genericity fails spectacularly.

The benefit of studying structured hyperplane sections of a variety is that, often, such sections are easy to control and understand. The counterpoint to this benefit is that the more special a linear section is, the less likely it is to be degree, dimension, or irreducible-generic. Incredibly, the hollow linear space of codimension 4 strikes a perfect balance for SO(4) between genericity and speciality. It allows us to search through the space of codimension-two linear sections of HSO(4) for a pair of hyperplanes which cut HSO(4) down to 40 *real* points. Here are two:

$$H_1 : \begin{bmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \cdot X = 0 \qquad H_2 : \begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \cdot X = 0,$$

where $\cdot$ denotes the entry-wise inner product. The 40 real intersection points are

$$\begin{bmatrix} 0 & -a & a & a \\ -a & 0 & -a & a \\ a & a & 0 & a \\ -a & a & a & 0 \end{bmatrix} \begin{bmatrix} 0 & a & -a & a \\ a & 0 & a & a \\ -a & -a & 0 & a \\ -a & a & a & 0 \end{bmatrix} \begin{bmatrix} 0 & -a & a & a \\ -a & 0 & a & -a \\ -a & a & 0 & a \\ a & a & a & 0 \end{bmatrix} \begin{bmatrix} 0 & a & -a & a \\ -a & 0 & a & a \\ a & a & 0 & -a \\ -a & -a & a & 0 \end{bmatrix} \begin{bmatrix} 0 & a & a & -a \\ a & 0 & a & a \\ -a & a & 0 & a \\ -a & -a & a & 0 \end{bmatrix} \begin{bmatrix} 0 & a & a & a \\ -a & 0 & -a & a \\ -a & a & 0 & -a \\ -a & -a & a & 0 \end{bmatrix} \begin{bmatrix} 0 & -a & -a & -a \\ a & 0 & -a & a \\ a & a & 0 & -a \\ a & -a & a & 0 \end{bmatrix} \begin{bmatrix} 0 & -a & -a & a \\ a & 0 & -a & -a \\ -a & -a & 0 & -a \\ a & -a & a & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

where $a = \pm\sqrt{1/3}$. Note that each matrix in the first row counts for two points, so $40 = 2{\cdot}8{+}8{+}8{+}8$. This resolves [BBB+17, Conjecture 7.1] for $n = 4$. We state our result for completeness.

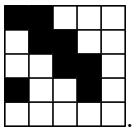
Theorem 4.1. There exists a witness set of SO(4) consisting of 40 real points.

This slice was found through the desire for the intersection points to hold combinatorial meaning. Specifically, we considered hyperplanes containing fixed point free signed permutation matrices, other than those in Figure 3. The slice above was the most elegant one we found.

5. Extending to SO(5)

One may hope that the hollow version of $\mathrm{SO}(n)$ decomposes in this beautiful way for $n > 4$. This is not the case: the variety of hollow 5×5 special orthogonal matrices is irreducible. Instead, we seek a different zero pattern for SO(5). For $n = 4$ the hollow pattern cuts the dimension of SO(4) from six to two. This is the rank of SO(4), that is, the dimension of its maximal torus. By an exhaustive calculation of all zero patterns with $\dim(\mathrm{SO}(5)) - \mathrm{rank}(\mathrm{SO}(5)) = 10 - 2 = 8$ zeros, modulo symmetry, we found there is a *unique* dimension-generic and degree-generic pattern:

($\star$)



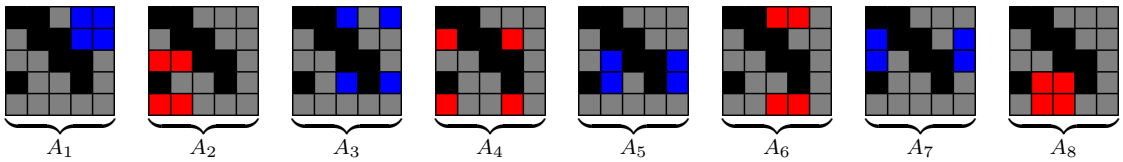
In fact, the exhaustive search shows that there is no other such pattern over the complex numbers either. We write $\mathrm{SO}^\star(5)$ for the set of 5×5 special orthogonal matrices with the zero pattern ($\star$). Just like HSO(4), the space $\mathrm{SO}^\star(5)$ decomposes dramatically. We write $\mathrm{SO}^\circ(3)$ for a copy of SO(3) with a single zero in some entry.

Theorem 5.1. The variety $\mathrm{SO}^\star(5) \subseteq \mathbb{R}^{25}$ is the union of 64 irreducible surfaces: 32 tori of degree four and 32 copies of $\mathrm{SO}^\circ(3) \times \mathrm{SO}(1) \times \mathrm{SO}(1)$ of degree eight. These 64 components may be grouped into 16 quadruples based on their zero pattern. There exists a bijection between these 16 quadruples and the facets of the polytope $\mathcal{P}$ (illustrated on the right of Figure 1) satisfying the following:

(1) The eight quadruples of tori correspond to the quadrilaterals with two degree four vertices.
(2) The eight quadruples of $\mathrm{SO}^\circ(3) \times \mathrm{SO}(1) \times \mathrm{SO}(1)$ correspond to the other facets.
(3) Quadruples intersect in curves if and only if their corresponding facets share an edge.
(4) Quadruples intersect in isolated points if and only if their facets meet in a vertex.
(5) Curve intersections of components are copies of $\mathrm{SO}(2) \times \mathrm{SO}(1) \times \mathrm{SO}(1) \times \mathrm{SO}(1) \cong \mathbb{S}^1$
(6) Point intersections of components are signed permutation matrices of determinant one.

The relevant bijection is illustrated in Figure 4.

5.1. **The thirty-two tori and their intersections.** Each torus in $\mathrm{SO}^\star(5)$ appears as a $2+2+1$ embedding of $\mathrm{SO}(2) \times \mathrm{SO}(2) \times \mathrm{SO}(1)$ into ($\star$). We identify the eight 2×2 blocks in ($\star$):



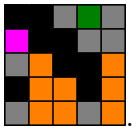
To combine two of these 2×2 blocks and a 1×1 block into a $2+2+1$ block matrix one must superimpose two cyclically adjacent (modulo 8) patterns to obtain a copy of $\mathrm{O}(2) \times \mathrm{O}(2) \times \mathrm{O}(1)$. Fixing a sign choice for each 2×2 block, we obtain tori, like T_{12+-} which is parametrized as

$$((x_1, x_2), (y_1, y_2)) \in \mathbb{S}^1 \times \mathbb{S}^1 \mapsto \begin{bmatrix} 0 & 0 & 0 & x_1 & -x_2 \\ 0 & 0 & 0 & x_2 & x_1 \\ y_1 & y_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ y_2 & -y_1 & 0 & 0 & 0 \end{bmatrix}.$$

There are eight possible block patterns, and four sign patterns for each, giving 32 tori:

$$\begin{array}{cccccccc} T_{12++} & T_{23++} & T_{34++} & T_{45++} & T_{56++} & T_{67++} & T_{78++} & T_{81++} \\ T_{12+-} & T_{23+-} & T_{34+-} & T_{45+-} & T_{56+-} & T_{67+-} & T_{78+-} & T_{81+-} \\ T_{12-+} & T_{23-+} & T_{34-+} & T_{45-+} & T_{56-+} & T_{67-+} & T_{78-+} & T_{81-+} \\ T_{12--} & T_{23--} & T_{34--} & T_{45--} & T_{56--} & T_{67--} & T_{78--} & T_{81--} \end{array}.$$

5.2. **The thirty-two sections of SO(3) and their intersections.** Each $\mathrm{O}^\circ(3) \times \mathrm{O}(1) \times \mathrm{O}(1)$ pattern appearing in $\mathrm{SO}^\star(5)$ can be uniquely identified by the positions of each $\mathrm{O}(1)$ in $(\star)$ since the complement of the columns/rows indexed by two such entries of $(\star)$ determines a 3×3 submatrix. For example, consider the pattern induced on $(\star)$ when $(1,4)$ and $(2,1)$ are ± 1:



Only the 3×3 submatrices of $(\star)$ which have 8 non-zero entries correspond to valid $3+1+1=5$ block patterns in $(\star)$. As was the case for the tori in $\mathrm{SO}^\star(5)$, such a pattern, together with two signs, identify a component of $\mathrm{SO}^\star(5)$ of this form. We will denote such components like $C_{(1,4)(2,1)++}$ to indicate the indices of ± 1's and their signs. This gives 32 components of degree $8 = \deg(\mathrm{SO}(3))$:

$$\begin{array}{cccccccc}
C_{(1,4)(2,1)++} & C_{(2,1)(3,2)++} & C_{(3,2)(4,3)++} & C_{(4,3)(1,4)++} & C_{(1,3)(2,4)++} & C_{(2,4)(3,1)++} & C_{(3,1)(4,2)++} & C_{(4,2)(1,3)++} \\
C_{(1,4)(2,1)+-} & C_{(2,1)(3,2)+-} & C_{(3,2)(4,3)+-} & C_{(4,3)(1,4)+-} & C_{(1,3)(2,4)+-} & C_{(2,4)(3,1)+-} & C_{(3,1)(4,2)+-} & C_{(4,2)(1,3)+-} \\
C_{(1,4)(2,1)-+} & C_{(2,1)(3,2)-+} & C_{(3,2)(4,3)-+} & C_{(4,3)(1,4)-+} & C_{(1,3)(2,4)-+} & C_{(2,4)(3,1)-+} & C_{(3,1)(4,2)-+} & C_{(4,2)(1,3)-+} \\
C_{(1,4)(2,1)--} & C_{(2,1)(3,2)--} & C_{(3,2)(4,3)--} & C_{(4,3)(1,4)--} & C_{(1,3)(2,4)--} & C_{(2,4)(3,1)--} & C_{(3,1)(4,2)--} & C_{(4,2)(1,3)--}
\end{array}.$$

We have now found all 64 components of $\mathrm{SO}^\star(5)$ since

$$32 \cdot 4 + 32 \cdot 8 = 384 = \deg(\mathrm{SO}(5)).$$

The bijection of Theorem 5.1 is given in Figure 4.

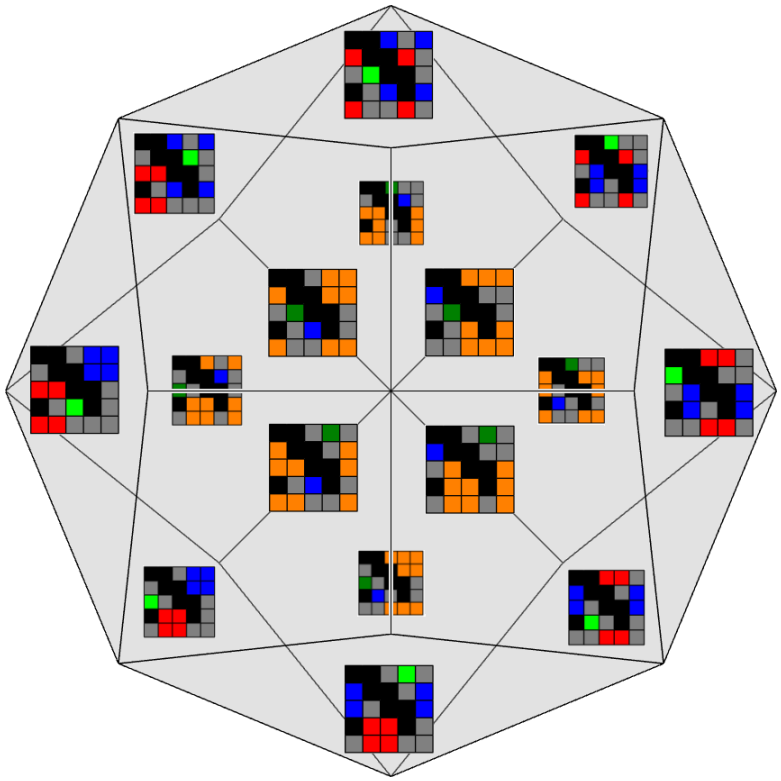
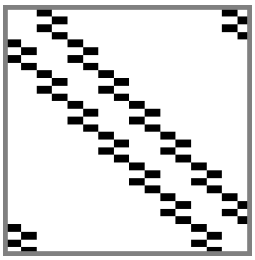


FIGURE 4. The polytope $\mathcal{P}$ with facets labeled by diagrams identifying quadruples of components of $\mathrm{SO}^\star(5)$ with the same zero pattern, giving the relevant bijection for Theorem 5.1.

5.3. **Two hundred fifty-six circles.** The curve intersection incidence matrix of the 32 tori is

$$\begin{bmatrix} 0 & \mathcal{I} & 0 & 0 & 0 & 0 & 0 & \mathcal{I} \\ \mathcal{I} & 0 & \mathcal{I} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{I} & 0 & \mathcal{I} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{I} & 0 & \mathcal{I} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{I} & 0 & \mathcal{I} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{I} & 0 & \mathcal{I} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{I} & 0 & \mathcal{I} \\ \mathcal{I} & 0 & 0 & 0 & 0 & 0 & \mathcal{I} & 0 \end{bmatrix} =$$

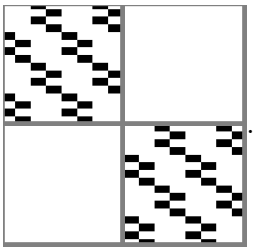


where $\mathcal{I} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

Essentially, these components intersect exactly when they share a 2×2 nonzero block and there is a sign agreement. Thus, all intersections are copies of $\mathrm{SO}(2) \times \mathrm{SO}(1) \times \mathrm{SO}(1) \times \mathrm{SO}(1)$.

Two $\mathrm{SO}^\circ(3) \times \mathrm{SO}(1) \times \mathrm{SO}(1)$ components intersect in a curve if and only if they share a 1×1 block and the respective signs of that entry agree. The intersection incidence matrix is

$$\begin{bmatrix} 0 & \mathcal{I} & 0 & \mathcal{I} & 0 & 0 & 0 & 0 \\ \mathcal{I} & 0 & \mathcal{I} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{I} & 0 & \mathcal{I} & 0 & 0 & 0 & 0 \\ \mathcal{I} & 0 & \mathcal{I} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{I} & 0 & \mathcal{I} \\ 0 & 0 & 0 & 0 & \mathcal{I} & 0 & \mathcal{I} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{I} & 0 & \mathcal{I} \\ 0 & 0 & 0 & 0 & \mathcal{I} & 0 & \mathcal{I} & 0 \end{bmatrix} =$$

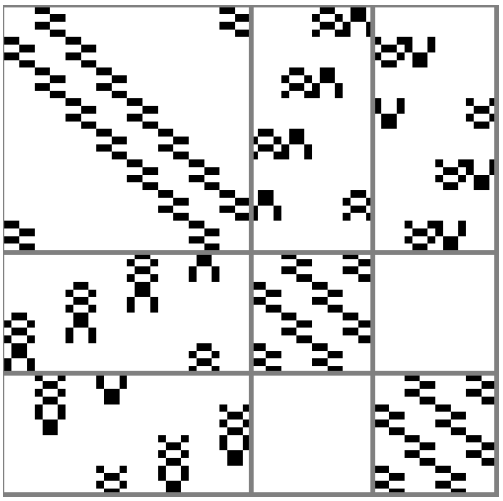


To complete the incidence picture of the curve intersections of the 64 components of $\mathrm{SO}^*(5)$, we need only establish how the tori intersect the $\mathrm{SO}^\circ(3) \times \mathrm{SO}(1) \times \mathrm{SO}(1)$ components. This can be done by hand using the combinatorics of the corresponding zero patterns, or via a direct computation in a computer algebra system, see [BG26]. For completeness, we provide the entire incidence matrix below in terms of $\mathcal{I}$ and the four matrices

$$J_1 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \qquad J_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \qquad J_3 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \qquad J_4 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

The complete curve intersection incidence matrix, along with its heatmap, is given below.

$$\left[\begin{array}{cccccccc||cccc|cccc} 0 & \mathcal{I} & 0 & 0 & 0 & 0 & 0 & \mathcal{I} & 0 & 0 & J_1 & J_2 & 0 & 0 & 0 & 0 \\ \mathcal{I} & 0 & \mathcal{I} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & J_3 & J_4 & 0 & 0 \\ 0 & \mathcal{I} & 0 & \mathcal{I} & 0 & 0 & 0 & 0 & 0 & J_1 & J_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{I} & 0 & \mathcal{I} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & J_4 & 0 & 0 & J_3 \\ 0 & 0 & 0 & \mathcal{I} & 0 & \mathcal{I} & 0 & 0 & J_1 & J_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{I} & 0 & \mathcal{I} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & J_3 & J_4 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{I} & 0 & \mathcal{I} & J_2 & 0 & 0 & J_1 & 0 & 0 & 0 & 0 \\ \mathcal{I} & 0 & 0 & 0 & 0 & 0 & \mathcal{I} & 0 & 0 & 0 & 0 & 0 & 0 & J_3 & J_4 & 0 \\ \hline\hline 0 & 0 & 0 & 0 & J_1 & 0 & J_2 & 0 & 0 & \mathcal{I} & 0 & \mathcal{I} & 0 & 0 & 0 & 0 \\ 0 & 0 & J_1 & 0 & J_2 & 0 & 0 & 0 & \mathcal{I} & 0 & \mathcal{I} & 0 & 0 & 0 & 0 & 0 \\ J_1 & 0 & J_2 & 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{I} & 0 & \mathcal{I} & 0 & 0 & 0 & 0 \\ J_2 & 0 & 0 & 0 & 0 & 0 & J_1 & 0 & \mathcal{I} & 0 & \mathcal{I} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & J_3 & 0 & J_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{I} & 0 & \mathcal{I} \\ 0 & J_4 & 0 & 0 & 0 & 0 & 0 & J_3 & 0 & 0 & 0 & 0 & \mathcal{I} & 0 & \mathcal{I} & 0 \\ 0 & 0 & 0 & 0 & 0 & J_3 & 0 & J_4 & 0 & 0 & 0 & 0 & 0 & \mathcal{I} & 0 & \mathcal{I} \\ 0 & 0 & 0 & J_3 & 0 & J_4 & 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{I} & 0 & \mathcal{I} & 0 \end{array}\right] =$$



Each $\mathcal{I}, J_1, J_2, J_3, J_4$ involves eight ones. In total, this accounts for 512 one's, representing 256 intersections. Each has the form of a circle realized as $\mathrm{SO}(2) \times \mathrm{SO}(1) \times \mathrm{SO}(1) \times \mathrm{SO}(1)$.

5.4. **Two hundred eighty-eight signed permutation matrices.** There are 288 unique zero-dimensional intersections of these 64 components. Continuing the scheme of grouping elements by their zero patterns, these zero-dimensional intersections exhibit 18 permutation matrix patterns; note that $288 = 2^4 \cdot 18$. These 18 permutation matrices are in bijection with the 18 vertices of $\mathcal{P}$ in Figure 5 according to how they appear as intersections of the components.

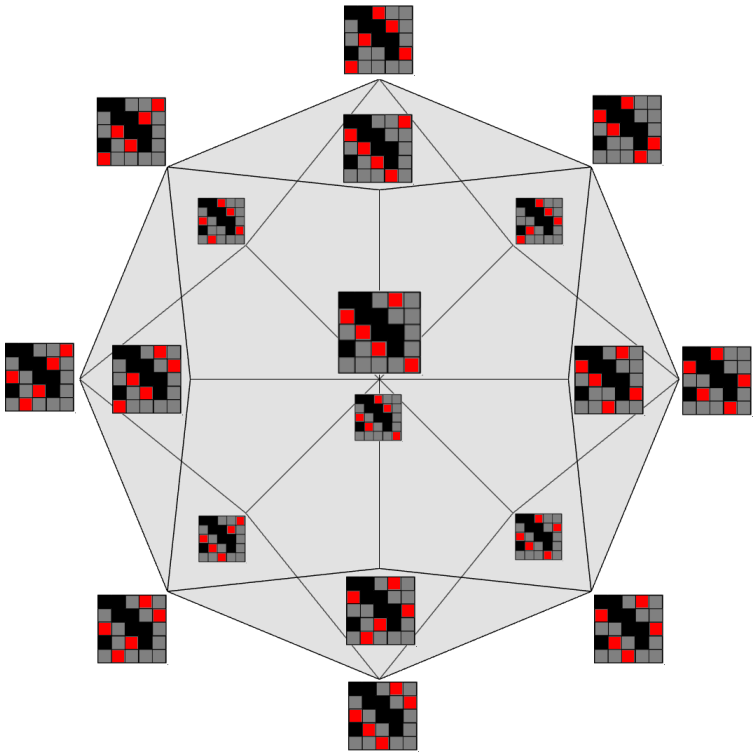


FIGURE 5. The vertices of $\mathcal{P}$ correspond to the zero-dimensional intersections of components of $\mathrm{SO}^\star(5)$. Each represents 2^4 signed permutation matrices.

These 18 vertices come in three flavours. These differences correspond to distinct cycle types.

(1) Eight three-way intersections of components corresponding to the degree three vertices. Each is a 5-cycle.
(2) Two four-way intersections of $\mathrm{SO}^\circ(3) \times \mathrm{SO}(1) \times \mathrm{SO}(1)$ components. They have cycle types $(4,1)$ and $(2,2,1)$.
(3) Eight intersections of one $\mathrm{SO}^\circ(3) \times \mathrm{SO}(1) \times \mathrm{SO}(1)$ and three torus components. Each has cycle type $(3,2)$.

Each permutation σ is subject to the condition that $\sigma(i) \notin \{i, i+1 \bmod 4\}$ due to the zero pattern ($\star$). Unlike in $\mathrm{HSO}(4)$, these are *all* of the permutation matrices which fit in the pattern ($\star$).

5.5. **A totally real witness set of $\mathbf{SO}(5)$.** We obtained a totally real witness set for $\mathrm{SO}(4)$ by choosing a special linear space through a subset of signed permutation matrices. In the case of $\mathrm{SO}(5)$, all of the signed permutation matrices compatible with the non-zero pattern ($\star$) appear as intersections of components of $\mathrm{SO}^\star(5)$. In particular, each such point is singular and so a slice through any such permutation matrix would not be degree-generic. We leave the problem of using the $\mathrm{SO}^\star(5)$ decomposition to find a totally real witness set to future research.

Conjecture 5.2. There exists a codimension-two linear slice of $\mathrm{SO}^\star(5)$ which is totally real.

6. Closing comments

6.1. **A word on smaller orthogonal groups.** We began with $n = 4$ because the cases of SO(1) and SO(2) are trivial and SO(3) is almost trivial. The dimensions of SO(1) and SO(2) equal their rank. Thus, the empty zero pattern realizes each as a variety of dimension equal to its rank, in analogy with our previous results. The set of 3×3 special orthogonal matrices forms a three-dimensional variety of degree eight. Up to symmetries, there is a unique degree-generic coordinate slice of codimension two, namely, the one where two coordinates are set to zero which do not share a row or column. Such a slice $\{x_{11} = x_{22} = 0\}$ decomposes SO(3) into four circles:

$$X_{12+} : \begin{bmatrix} 0 & 1 & 0 \\ x_{21} & 0 & x_{23} \\ x_{31} & 0 & x_{33} \end{bmatrix}, \qquad X_{12-} : \begin{bmatrix} 0 & -1 & 0 \\ x_{21} & 0 & x_{23} \\ x_{31} & 0 & x_{33} \end{bmatrix}, \qquad X_{21+} : \begin{bmatrix} 0 & x_{12} & x_{13} \\ 1 & 0 & 0 \\ 0 & x_{31} & x_{33} \end{bmatrix}, \qquad X_{21-} : \begin{bmatrix} 0 & x_{12} & x_{13} \\ -1 & 0 & 0 \\ 0 & x_{31} & x_{33} \end{bmatrix}.$$

The intersection incidence graph of these four curves is the polygon with four edges, with $X_{ij\pm}$ on opposite sides. The four vertices correspond to the transposition $(1, 2)$ with signs.

6.2. **The necessity of the quadruples.** The beautiful story for SO(4) struggles to extend to SO(5): although there exists a unique zero pattern ($\star$) which decomposes SO(5) into many components of dimension $\text{rank}(\text{SO}(5)) = 2$, this pattern is more mysterious than simply being hollow, and one must combine components to encode the incidence structure in the face lattice of a polytope.

Proposition 6.1. There does not exist a 3-polytope whose 64 facets are in bijection with the components of $\text{SO}^\star(5)$ such that its facets intersect in an edge if and only if the corresponding components intersect in a curve.

Proof. The incidence matrix of the 64 components has 512 ones, and thus such a polytope would have 256 edges. By Euler's formula, the number of vertices would have to be $194 = 256 - 64 + 2$. If every vertex had degree at least three then there would be at least $\frac{194 \cdot 3}{2} = 291$ edges. So some vertex has degree less than three, a contradiction to it being a vertex of a 3-polytope. □

6.3. **Extending to SO(6).** For $n = 6$, we have a result in the negative direction. Continuing the pattern for $n = 4, 5$, we seek a zero pattern which is degree and dimension-generic and cuts SO(6) to dimension three, that of its maximal torus. We hope such a pattern exists which decomposes SO(6) significantly. To find such a slice, one must identify a corresponding pattern of 12 zeros.

Theorem 6.2. There is no pattern of 12 zeros in a 6×6 matrix which represents a slice of codimension 12 that is degree and dimension-generic with respect to SO(6).

Proof. This is proven by exhaustive computation over all $(6, 6)$-bipartite graphs with 12 edges and no vertex of degree six or higher. The computation first collects all 5816 such graphs up to symmetry using Nauty [MP13]. For each, we symbolically compute the dimension and degree of the corresponding slice in Macaulay2 [GS26]; no dimension-generic and no degree-generic section arises (even over complex numbers). The full results are available at [BG26]. □

Theorem 6.2 tells us that generalizing what is done in this paper to $n = 6$ is impossible if one hopes for too many of the properties of the previous cases to persist. An alternative approach is to interpret the zero pattern as affine spans of signed permutation matrices. Generalizing from this point of view requires deep knowledge of the *algebraic matroid* of signed permutation matrices.

6.4. **The polytope $\mathcal{P}$.** As far as the authors are aware, the polytope $\mathcal{P}$ illustrated on the right of Figure 1 has never been identified in a research article. Like the Johnson solid J17, our polytope $\mathcal{P}$ is comprised of two halves: the near and far halves of the right of Figure 1. In both J17 and $\mathcal{P}$, these halves are twisted by 45 degrees. However, $\mathcal{P}$ does not have an edge-cycle in a hyperplane equator (like a bicupola) nor a belt (like an elongated bicupola). Our polytope has the same f-vector as J28 and J29. Moreover, the combinatorial symmetries of J29 and $\mathcal{P}$ are both D_{16}. Nevertheless, our polytope $\mathcal{P}$ is combinatorially distinct from all of these examples.

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