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Tilings of the sphere by congruent pentagons IV: Edge combination a^4b with general angles

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Abstract

We classify edge-to-edge tilings of the sphere by congruent pentagons with the edge combination a^4b and with any irrational angle in degrees: they are three 1-parameter families of pentagonal subdivisions of the Platonic solids, with $12, 24$ and 60 tiles; and a sequence of 1-parameter families of pentagons admitting non-symmetric 3-layer earth map tilings together with their various rearrangements under extra conditions. Their parameter moduli and geometric data are all computed in both exact and numerical form. The total numbers of different tilings for any fixed such pentagon are counted explicitly. As a byproduct, the degenerate pentagons produce many new non-edge-to-edge quadrilateral tilings.

Keywords: spherical tiling, almost equilateral pentagon, pentagonal subdivision, earth map tiling, irrational angle.
2020 Mathematics subject classification: Primary 52C20, 05B45.

1 Introduction

A lot of progress has been made on spherical tilings in recent years. Non-edge-to-edge multihedral tilings of the sphere by regular polygons have been classified in [2]. Edge-to-edge monohedral quadrilateral tilings of the sphere have been fully classified in [4, 9, 10, 11]. In an edge-to-edge tiling of the sphere by congruent pentagons, the pentagon can have five possible edge combinations [12, Lemma 9]: $a^2b^2c, a^3bc, a^3b^2, a^4b, a^5$. Tilings for $a^2b^2c, a^3bc, a^3b^2, a^5$ were classified in [3, 8, 12, 13], whose methods were further extended by Cheung, Luk and Yan to classify a^4b-tilings in a preprint [5] of **174 pages**. Our strategy was developed independently and does not rely on their case reductions, reconstruction arguments or classification proof. Agreement between the two classifications provides an independent verification.

The present paper treats pentagons with at least one irrational angle in degrees; the sequel [7] treats the rational-angle case using number-theoretic

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methods. This totally new strategy was designed for this last and the most difficult a^4b-pentagon case originally, though our papers applying this strategy to quadrilateral case [10, 11] were completed earlier since the trigonometric Diophantine equations for pentagons are much harder to solve.

Together, our two papers give a shorter proof ($45 + 23 =$ **68 pages**) and provide four new results:

1. full geometric data for all prototiles;
2. 3D illustrations of every tiling type; (https://github.com/spherical1tiling2)
3. explicit counts of the tilings admitted by each prototile;
4. new non-edge-to-edge quadrilateral tilings obtained from degenerate pentagons.

An a^4b-pentagon is given by Figure 1, with normal edge a, thick edge b and angles α, β, γ, δ, ϵ as indicated. The second picture is the mirror image or flip of the first. The angles determine the orientation. Conversely, the edge lengths and the orientation also determine the angles. So we may present the tiling by shading instead of indicating all angles. Throughout this paper, an a^4b-tiling is always an edge-to-edge tiling of the sphere by congruent simple pentagons in Figure 1, such that all vertices have degree ≥ 3. We call such a pentagon **almost equilateral**. We assume $a \neq b$ and allow straight angles.

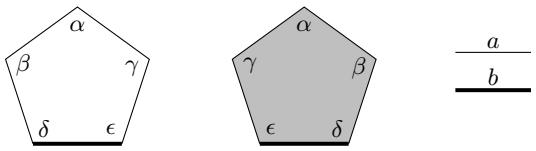


Figure 1: Pentagons with the edge combinations a^4b.

We use $\alpha^{n_1}\beta^{n_2}\gamma^{n_3}\delta^{n_4}\epsilon^{n_5}$ to mean a vertex having n_1 copies of α, n_2 copies of β, etc. The anglewise vertex combination(s), abbreviated as **AVC**, is the collection of all vertices in a tiling. Then the notation $T(16\alpha\delta\epsilon, 8\beta^2\gamma, 2\gamma^4)$ means the tiling has exactly 16 vertices $\alpha\delta\epsilon$, 8 vertices $\beta^2\gamma$, and 2 vertices γ^4, and is *uniquely* determined by them. In general there may exist several different tilings with the same set of vertices. Then we use $\{4\alpha\beta\gamma, 14\alpha\delta\epsilon, 6\beta^2\gamma, 4\beta\delta\epsilon, 2\alpha\gamma^3, 2\gamma^2\delta\epsilon : 3\}$ to mean that there are 3 different tilings.

When an almost equilateral pentagon is symmetric about an axis through α, i.e. $\beta = \gamma$, $\delta = \epsilon$, it can be divided into two congruent quadrilaterals along this axis, and the corresponding classification has already been completed by [9, 10, 11]. We translate it as the following theorem.

Theorem 1. *All symmetric a^4b-tilings are:*

1. *A 1-parameter family of symmetric a^4b-pentagonal subdivisions of the tetrahedron with 12 tiles $T(12\alpha\delta\epsilon, 4\beta^3, 4\gamma^3)$ (see the first picture of Figure 4 and Table 1).*

2. *A sequence of unique symmetric a^4b-pentagons with five angles* $(\frac{8}{f}, 1-\frac{4}{f}, 1-\frac{4}{f}, \frac{1}{2}+\frac{2}{f}, \frac{1}{2}+\frac{2}{f})\pi$ *and*

$$\cos a = 1 - 2\left(\frac{\sqrt{5}-1}{4\cos\frac{4\pi}{f}}\right)^2, \quad \cos b = 2\left(\frac{(3-\sqrt{5})\cos^2\frac{4\pi}{f}+\sqrt{5}-2}{\cos\frac{4\pi}{f}}\right)^2 - 1,$$

each admitting a symmetric 3*-layer earth map tiling* $T(2m\,\alpha\beta\gamma, 2m\,\beta\delta\epsilon, 2m\,\gamma\delta\epsilon, 2\alpha^m)$ *by* $f = 4m$ *tiles for any* $m \geq 4$*, among which each odd* $m = 2k+1$ *case admits two standard flip modifications (see Figure 2):*
$T(4k\,\alpha\beta\gamma, (4k+2)\beta\delta\epsilon, (4k+2)\gamma\delta\epsilon, 2\alpha^{k+1}\beta, 2\alpha^{k+1}\gamma)$*, and*
$T((4k+2)\alpha\beta\gamma, (4k+1)\beta\delta\epsilon, (4k+1)\gamma\delta\epsilon, \alpha^{k+1}\beta, \alpha^{k+1}\gamma, 2\alpha^k\delta\epsilon)$.

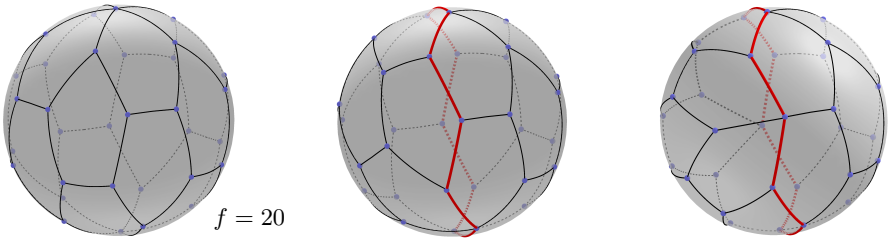


Figure 2: A symmetric 3-layer earth map tiling and its two standard flip modifications.

The pentagonal subdivision introduced in [12] replaces each triangular face of the regular tetrahedron, octahedron or icosahedron by three congruent pentagons determined by a freely chosen point V (see Figure 3). The exact range of V was studied in great detail in [8]. Only when V is on the perpendicular bisector of AB will the pentagon become almost equilateral (see Figure 4). The parameter t in Table 1 represents the signed spherical distance of V to AB.

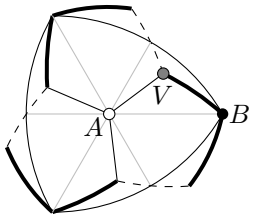


Figure 3: Pentagonal subdivision of a regular triangle face.

Henceforth, we only focus on non-symmetric a^4b-tilings. We will prove:

Theorem 2. *All non-symmetric* a^4b*-tilings with general angles are:*

1. *Two* 1*-parameter families of pentagonal subdivisions of the octahedron and icosahedron with* 24 *tiles* $T(24\alpha\delta\epsilon, 8\gamma^3, 6\beta^4)$ *and* 60 *tiles* $T(60\alpha\delta\epsilon, 20\gamma^3, 12\beta^5)$ *(see the second and third pictures of Figure 4 and Table 1).*

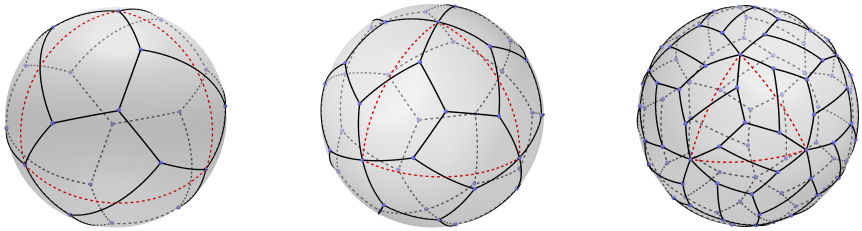


Figure 4: Pentagonal subdivision of the tetra-, octa- and icosa-hedron.

Table 1: The geometric data of three 1-parameter families of a^4b-pentagonal subdivisions.

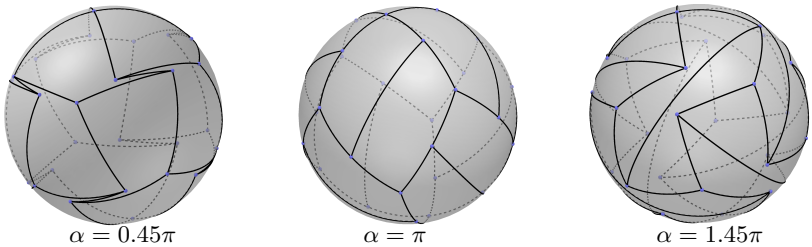
Vertices and tilings	α, β, γ, δ, ϵ, a, b
$T(12\alpha\delta\epsilon,\ 4\beta^3,\ 4\gamma^3)$	$\beta=\gamma=\frac{2\pi}{3}$, $\cos a=\frac{\sqrt{6}\cos t}{3}$, $b=\frac{\pi}{2}-2t$, $\cos\frac{\alpha}{2}=\frac{\sqrt{2}\sin t}{\sqrt{1+2\sin^2 t}}$,
	$\cos\epsilon=\frac{1-\sqrt{2}\cos t\cos\frac{b}{2}}{\sqrt{1+2\sin^2 t\,\sin\frac{b}{2}}}$, $\delta=2\pi-\alpha-\epsilon$, $t\in(-\frac{\pi}{4},\frac{\pi}{4})\backslash\{t_0\}$, $t_0=\arctan\frac{\sqrt{5}}{5}$. The pentagon is equilateral at t_0.
$T(24\alpha\delta\epsilon,\ 8\gamma^3,\ 6\beta^4)$	$\beta=\frac{\pi}{2}$, $\gamma=\frac{2\pi}{3}$, $\cos a=\frac{\sqrt{18+6\sqrt{3}}\cos t}{6}$, $\cos\frac{b}{2}=\frac{1}{12}(\sqrt{3}+\sqrt{3}(\sqrt{(4-2\sqrt{3})\cos^2 t+4}\cos t)+6\sin t)$,
	$\cos\frac{\alpha}{2}=\frac{\sqrt{3+\sqrt{3}}\sin t}{\sqrt{6-(3+\sqrt{3})\cos^2 t}}$, $\cos\epsilon=\frac{2-\sqrt{3+\sqrt{3}}\cos t\cos\frac{b}{2}}{\sqrt{6-(3+\sqrt{3})\cos^2 t\sin\frac{b}{2}}}$, $\delta=2\pi-\alpha-\epsilon$,
	$t\in(t_1,t_2)\backslash\{t_0\}$, $\tan t_1=-\frac{3\sqrt[4]{3}\sqrt{2}}{\sqrt{18+6\sqrt{3}}}$, $\tan t_2=\frac{\sqrt{6-2\sqrt{3}}}{2}$, $\cos t_0$ is the largest real root of
	$(150+75\sqrt{3})x^4-\sqrt{18+6\sqrt{3}}(6+10\sqrt{3})x^3-(90+78\sqrt{3})x^2+12\sqrt{18+6\sqrt{3}}x+18=0$, $t_0\approx 0.0892\pi$.
$T(60\alpha\delta\epsilon,\ 20\gamma^3,\ 12\beta^5)$	$\beta=\frac{2\pi}{5}$, $\gamma=\frac{2\pi}{3}$, $\cos a=\frac{\sqrt{450+30\sqrt{75+30\sqrt{5}}}\cos t}{30}$,
	$\cos\frac{b}{2}=\frac{2^{\frac{5}{4}}(\sqrt{5}+1)\sqrt{3\sqrt{10-2\sqrt{5}}+\sqrt{15}+\sqrt{3}}\cos t+(\sqrt{5}-1)(2^{\frac{1}{4}}\sqrt{3\sqrt{10-2\sqrt{5}}-\sqrt{15}-\sqrt{3}}\cos t+6(5-\sqrt{5})^{\frac{1}{4}}\sin t)}{24(5-\sqrt{5})^{\frac{1}{4}}}$,
	$\cos\frac{\alpha}{2}=\frac{\sqrt{6\sqrt{5-\sqrt{5}}+\sqrt{30}+\sqrt{6}}\sin t}{\sqrt{6\sqrt{5-\sqrt{5}}(1+\sin^2 t)-(\sqrt{30}+\sqrt{6})\cos^2 t}}$, $\cos\epsilon=\frac{(\sqrt{5}+1)(5-\sqrt{5})^{\frac{1}{4}}-\sqrt{6\sqrt{5-\sqrt{5}}+\sqrt{30}+\sqrt{6}}\cos t\cos\frac{b}{2}}{\sqrt{(6+6\sin^2 t)\sqrt{5-\sqrt{5}}-(\sqrt{30}+\sqrt{6})\cos^2 t\sin\frac{b}{2}}}$, $\delta=2\pi-\alpha-\epsilon$,
	$t\in(t_1,t_2)\backslash\{t_0\}$, $\tan t_0=\frac{\sqrt{3}}{1513800}((\sqrt{75+30\sqrt{5}}(42050-16820\sqrt{5})+25230\sqrt{5})\sqrt{90+6\sqrt{75+30\sqrt{5}}}$
	$+3((11\sqrt{5}-27)\sqrt{75+30\sqrt{5}}-5\sqrt{5}-35)\sqrt{\sqrt{75+30\sqrt{5}}(7259325+3566635\sqrt{5})-36362925\sqrt{5}-70401375})$,
	$\tan t_1=\frac{(\sqrt{5}-3)}{4}(\sqrt{15+6\sqrt{5}}+\sqrt{75+30\sqrt{5}}-3\sqrt{5}-7)^{\frac{1}{2}}$, $\tan t_2=\frac{\sqrt{600-10\sqrt{6}(5+\sqrt{5})^{\frac{3}{2}}}}{20}$.

2. *A sequence of 1-parameter families of pentagons (whose geometric data are given in Table 2 and shown in Figure 5) each admitting a non-symmetric 3-layer earth map tiling* $T(4m\,\alpha\delta\epsilon, 2m\,\beta^2\gamma, 2\gamma^m)$ *with* $4m$ *tiles for any* $m \geq 4$*, among which each odd* $m = 2k+1$ *case always admits a standard flip modification:* $T((8k+4)\alpha\delta\epsilon, 4k\,\beta^2\gamma, 4\beta\gamma^{k+1})$ *(see Figure 6). Moreover,*

   - *For* $f = 16$*, the unique pentagon with angles* $(\frac{\pi}{2}, \frac{3\pi}{4}, \frac{\pi}{2}, \delta, \frac{3\pi}{2} - \delta)$*,* $\cos\delta = \frac{1}{\sqrt[4]{2}}$*,* $\cos a = \sqrt{\sqrt{2}-1}$*,* $\cos b = \sqrt{2}\sqrt{\sqrt{2}-1}$*, admits a unique flip modification:* $T(8\alpha\delta\epsilon, 8\alpha\beta^2, 8\gamma\delta\epsilon, 2\gamma^4)$ *(see Figure 7).*
   - *For* $f = 8k+4$ *(i.e.* $m = 2k+1$*) with any* $k \geq 2$*, the unique pentagon determined by* $\alpha = \beta = \pi - \frac{4\pi}{f}$ *(in Table 2), admits a lot more modifications (shown in Table 3 and Figure 8, 9).*

Table 2: A sequence of 1-parameter families of pentagons admitting non-symmetric 3-layer earth map tilings.

$\beta = \pi - \frac{4\pi}{f}$, $\gamma = \frac{8\pi}{f}$, $f = 4m\,(m \geq 4)$
$p := \cos\alpha, q := \cos\frac{4\pi}{f}$,
$\cos\delta = \sqrt{\frac{-4\sqrt{p^2(p+1)(p-1)^2(2q-1)^2(q-1)^3(pq+p-q)}+(8p^3-8p^2-4p+4)q^3-(12p^3-16p^2-4p+4)q^2-(14p^2-p-1)q+4p^3+6p^2-p}{-8p^2q+8p^2+4q^2-4q+1}}$ $(\alpha \in [\frac{\pi}{2}, \pi])$,
$\cos\delta = \sqrt{\frac{4\sqrt{p^2(p+1)(p-1)^2(2q-1)^2(q-1)^3(pq+p-q)}+(8p^3-8p^2-4p+4)q^3-(12p^3-16p^2-4p+4)q^2-(14p^2-p-1)q+4p^3+6p^2-p}{-8p^2q+8p^2+4q^2-4q+1}}$ $(\alpha \in (\pi, \frac{3\pi}{2})$; or
$\alpha \in (2\arctan(\sqrt{-5+4\sqrt{2}}), \frac{\pi}{2})$ for $f = 16$, $\alpha \in (\arccos(\frac{\sqrt{2(1-2q)(2q^3-q^2-2q+1+\sqrt{(3q+1)(2q-1)^2(1-q)^3}}}{4q-2}), \frac{\pi}{2})$ for $f \geq 20$).
$\epsilon = 2\pi - \alpha - \delta$, $\cos a = \frac{\sin\frac{\alpha}{2}\sin\frac{2\pi}{f}\cos\frac{6\pi}{f}\cos\delta - \cos\alpha\sin\frac{4\pi}{f}\sin(\frac{\alpha}{2}+\delta)}{\sin\frac{\alpha}{2}\cos\frac{2\pi}{f}\cos\frac{6\pi}{f}\sin\delta}$, $\cos b$ is given in Lemma 12.
When $\alpha \in (\arccos(q-2q^2), \pi)$, the pentagon is convex.
$\lim_{f\to\infty}\beta = \pi$, $\lim_{f\to\infty}\gamma = 0$, $\lim_{f\to\infty}\delta = 0$, $\lim_{f\to\infty}\epsilon = (2\pi-\alpha) \in (\frac{\pi}{2}, \frac{3\pi}{2}]$, $\lim_{f\to\infty} b = 2\lim_{f\to\infty} a \in [\frac{\pi}{3}, \frac{\pi}{2}]$.
When $f = 16$ and $\alpha \approx 0.663\pi$, $\cos a = \cos b \approx 0.779$;
When $f = 20$ and $\alpha \approx 0.628\pi$, $\cos a = \cos b \approx 0.776$;
When $f = 24$ and $\alpha = 0.5\pi$, $\cos a = \cos b \approx 0.681$;
When $f = 24$ and $\alpha \approx 0.522\pi$, $\cos a = \cos b \approx 0.706$. (These 4 equilateral pentagons appeared in [3].)



$\alpha = 0.45\pi$ $\alpha = \pi$ $\alpha = 1.45\pi$

Figure 5: The 1-parameter family of non-symmetric 3-layer earth map tilings with $f = 16$ for three particular parameter values.

The geometric data in Table 2 depends on the angle parameter α, and Figure 5 shows some examples in the family with $f = 16$: α is close to the lower bound $2\arctan(\sqrt{-5+4\sqrt{2}}) \approx 0.43\pi$ in the first picture, is π in the second, and is

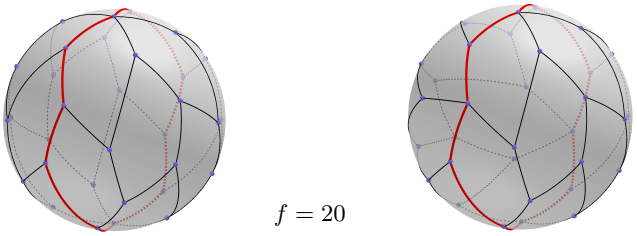


Figure 6: A generic non-symmetric 3-layer earth map tiling and its standard flip modification.

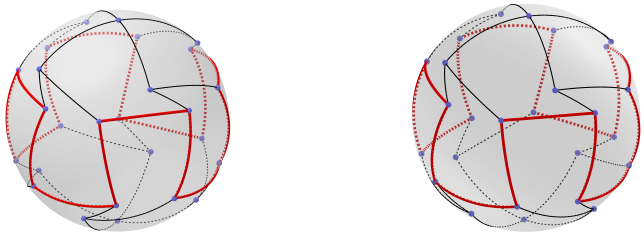


Figure 7: A special flip modification when $f = 16$ and $\alpha = \frac{\pi}{2} = \gamma$.

Table 3: Various modifications of the non-symmetric 3-layer earth map tiling with $\alpha = \beta = (1 - \frac{4}{f})\pi$, $f = 8k + 4$, $k \geq 2$ if both $\alpha\delta\epsilon, \beta\delta\epsilon$ appear.

	Vertices and tilings
AVC 1	$\{4\alpha\beta\gamma, (8k-2)\alpha\delta\epsilon, (4k-2)\beta^2\gamma,\ 4\beta\delta\epsilon, 2\alpha\gamma^{k+1}, 2\gamma^k\delta\epsilon : 3\}$
AVC 2	$T(2\alpha^2\gamma, 6\alpha\beta\gamma, (8k-6)\alpha\delta\epsilon, (4k-4)\beta^2\gamma, 6\beta\delta\epsilon, 4\gamma^k\delta\epsilon)$
AVC 3	$T(2\alpha^2\gamma, 4\alpha\beta\gamma, (8k-6)\alpha\delta\epsilon, (4k-4)\beta^2\gamma, 8\beta\delta\epsilon, 2\alpha\gamma^{k+1}, 2\gamma^k\delta\epsilon)$
AVC 4	$T(2\alpha^2\gamma, 5\alpha\beta\gamma, (8k-6)\alpha\delta\epsilon, (4k-4)\beta^2\gamma, 7\beta\delta\epsilon, \alpha\gamma^{k+1}, 3\gamma^k\delta\epsilon)$
AVC 5	$\{2\alpha^2\gamma, 6\alpha\beta\gamma, (8k-7)\alpha\delta\epsilon, (4k-5)\beta^2\gamma, 8\beta\delta\epsilon, \alpha\gamma^{k+1}, 3\gamma^k\delta\epsilon : 2k-2\}$
AVC 6	$\{4\alpha^2\gamma, 6\alpha\beta\gamma, (8k-10)\alpha\delta\epsilon, (4k-6)\beta^2\gamma, 10\beta\delta\epsilon, 4\gamma^k\delta\epsilon : 2\}$
AVC 7	$\{4\alpha^2\gamma, 7\alpha\beta\gamma, (8k-11)\alpha\delta\epsilon, (4k-7)\beta^2\gamma, 11\beta\delta\epsilon, 4\gamma^k\delta\epsilon : 3k-5\}$
AVC 8	$\{4\alpha^2\gamma, 8\alpha\beta\gamma, (8k-12)\,\alpha\delta\epsilon, (4k-8)\beta^2\gamma, 12\beta\delta\epsilon,\ 4\gamma^k\delta\epsilon : (2(k-1)(k-2) + \lfloor\frac{k}{2}\rfloor(k - \lfloor\frac{k}{2}\rfloor))\}$

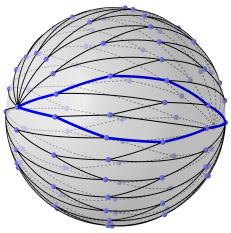


Figure 8: The only UFO for a tiling with AVC 1 and 60 tiles.

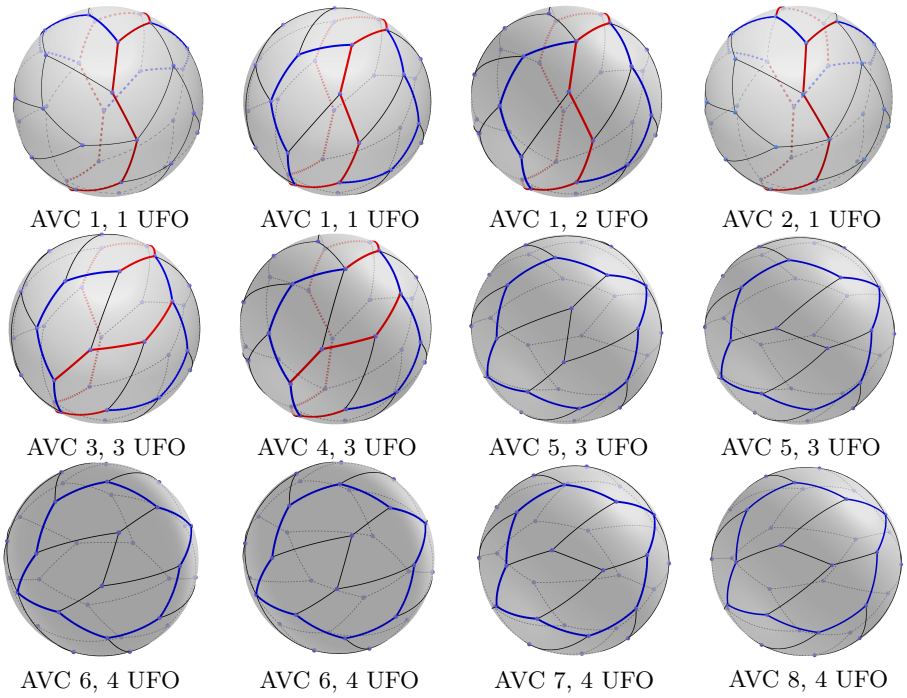


Figure 9: All modifications for the non-symmetric 3-layer earth map tiling with 20 tiles and $\alpha = \beta = \frac{4\pi}{5}$.

close to the upper bound $\frac{3\pi}{2}$ in the third. Note that the pentagon in Figure 7 corresponds to the parameter $\alpha = \frac{\pi}{2}$ in this family.

All tilings in Table 3 contain at least one special block called a UFO. Here we use UFO (Unidentified Flying Object) to denote the symmetric 4-tile patch bounded by the blue lines in Figure 8, since it looks like a flying saucer or spaceship. Any UFO contains two $\beta\delta\epsilon$ in the interior and is centrally symmetric together with two mirror symmetries. Figure 9 shows all tilings with $f = 20$ and 8 types of AVC in Table 3, where the number of UFOs is also counted for each tiling by playing with the 3D pictures in GeoGebra®. Note that only AVC $5, 7, 8$ in Table 3 will admit more different tilings as $f = 8k + 4$ becomes bigger. However there are still only 4 UFOs for all these tilings with AVC $7, 8$, and only 3 UFOs for all these tilings with AVC 5. For all other AVC in Table 3, the number of different tilings and the number of different UFOs are the same as Case $f = 20$ in Figure 9.

A sequel [7] of this paper will handle a^4b-pentagons with all angles being rational in degrees by solving some trigonometric Diophantine equations, to complete our full classification of edge-to-edge tilings of the sphere by congruent pentagons. It turns out that there are no more new prototiles, nor new tilings. Therefore, this paper, its sequel [7], and the earlier papers [12, 13, 3] together give all edge-to-edge spherical tilings by congruent pentagons. This is in sharp contrast to the quadrilateral case [10], where there is an abundance of tilings of quadrilaterals with all angles being rational.

Outline of the paper

The basic facts of tilings and basic techniques for complete classification are presented in Section 2, including some technical results specific to a^4b-tilings. We call a vertex with three a-edges an a^3**-vertex**, and a vertex with two a-edges and one b-edge an a^2b**-vertex**. Tilings with two different types of a^2b-vertices are studied in Section 3. Tilings with only one type of a^2b-vertex are studied in Section 4, 5 and 6. The non-edge-to-edge quadrilateral tilings induced from degenerate pentagonal tilings in this paper are summarized in the last section.

Acknowledgments

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2 Basic Facts

For the basics of tilings, we adopt the definitions and conventions in the books by Grünbaum and Shephard [6], and by Adams [1]. Angles are in units of π radians, except in Lemmas 11 and 12, where they are in radians. All side lengths

are in radians. So the sum of all angles at a vertex is 2. We present some basic facts and techniques in this section.

Let v, e, f be the numbers of vertices, edges, and tiles in a pentagonal tiling. Let v_k be the number of vertices of degree k. We have the following basic formulas in [12].

$$f = 12 + 2\sum_{k\geq 4}(k-3)v_k = 12 + 2v_4 + 4v_5 + 6v_6 + \cdots, \tag{2.1}$$

$$v_3 = 20 + \sum_{k\geq 4}(3k-10)v_k = 20 + 2v_4 + 5v_5 + 8v_6 + \cdots. \tag{2.2}$$

So $f \geq 12$ and $v_3 \geq f + 8 \geq 20$. The equality (2.2) shows that most vertices have degree 3.

Lemma 3 ([12, Lemma 4])**.** *If all tiles in a tiling of the sphere by f pentagons have the same five angles $\alpha, \beta, \gamma, \delta, \epsilon$, then*

$$\alpha + \beta + \gamma + \delta + \epsilon = 3 + \tfrac{4}{f},$$

ranging in $(3, \frac{10}{3}]$. In particular no vertex contains all five angles.

Lemma 4 ([12, Lemma 1])**.** *Any pentagonal tiling of the sphere has a "special" tile, such that four vertices have degree* 3 *and the fifth vertex has degree* 3, 4 *or* 5*.*

Lemma 5 (Parity Lemma, [13, Lemma 10])**.** *In an a^4b-tiling, the total number of ab-angles (i.e. δ and ϵ) at any vertex is even.*

In a tiling of the sphere by f congruent tiles, each angle of the tile appears f times in total. If one vertex has more α than β, there must exist another vertex with more β than α. Such global counting induces many interesting and useful results.

Lemma 6 (Balance Lemma, [11, Lemma 6])**.** *In an a^4b-tiling, if either $\delta^2\cdots$ or $\epsilon^2\cdots$ is not a vertex, then any vertex either has no δ, ϵ, or is of the form $\delta\epsilon\cdots$ with no more δ, ϵ in the remainder.*

Lemma 7. *In an a^4b-tiling, an a^3-vertex and an a^2b-vertex must both appear.*

Proof. By Lemma 4, we know that either $\delta\cdots$ or $\epsilon\cdots$ is an a^2b-vertex. If there is no a^3-vertex, then every degree 3 vertex is an a^2b-vertex. By the equations (2.1) and (2.2), we have $\#\delta + \#\epsilon \geq 2v_3 > 2f$ in the tiling, contradicting $\#\delta + \#\epsilon = 2f$. □

The very useful tool *adjacent angle deduction* (abbreviated as **AAD**) has been introduced in [12, Section 2.5]. We give a quick review here using Figure 10. Let "|" denote an a-edge and "**|**" denote a b-edge. Then we indicate the arrangements of angles and edges by denoting the vertices as $|\beta|\beta|\delta\mathbf{|}\delta|$. The notation can be reversed, such as $|\beta|\beta|\delta\mathbf{|}\delta| = |\delta\mathbf{|}\delta|\beta|\beta|$; and it can be rotated,

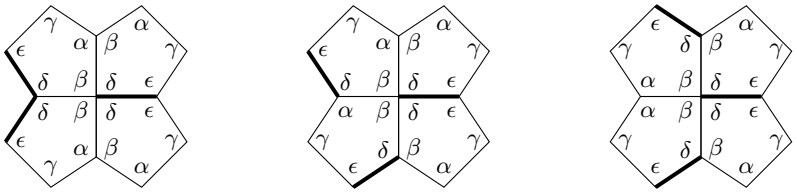


Figure 10: Different Adjacent Angle Deductions of $\beta^2\delta^2$.

such as $|\beta|\beta|\delta\rfloor\delta| = |\beta|\delta\rfloor\delta|\beta| = \rfloor\delta|\beta|\beta|\delta\rfloor$. We also denote the first vertex in Figure 10 as $\delta\rfloor\delta\cdots$, $|\delta\rfloor\delta|\cdots, \beta|\delta\cdots$, $|\delta\rfloor\delta|\beta|\cdots$, and denote the consecutive angle segments as $\delta\rfloor\delta$, $|\delta\rfloor\delta|$, $\beta|\delta$, $|\delta\rfloor\delta|\beta|$.

The pictures of Figure 10 have the same vertex $|\beta|\beta|\delta\rfloor\delta|$, but different arrangements of the four tiles. To indicate the difference, we write ${}^{\lambda}\theta^{\mu}$ to mean λ, μ are the two angles adjacent to θ in the quadrilateral. The first picture has the AAD $|^{\alpha}\beta^{\delta}|^{\delta}\beta^{\alpha}|^{\beta}\delta^{\epsilon}\rfloor^{\epsilon}\delta^{\beta}|$. The second and third have the AAD $|^{\alpha}\beta^{\delta}|^{\alpha}\beta^{\delta}|^{\beta}\delta^{\epsilon}\rfloor^{\epsilon}\delta^{\beta}|$ and $|^{\delta}\beta^{\alpha}|^{\alpha}\beta^{\delta}|^{\beta}\delta^{\epsilon}\rfloor^{\epsilon}\delta^{\beta}|$ respectively. The following useful lemma is from [12, Lemma 10].

Lemma 8 ([12, Lemma 8])**.** *Suppose an angle θ does not appear at degree 3 vertices in a tiling of the sphere by pentagons with the same angle combination. Then one of $\{\theta^4, \theta^3\rho, \theta^5\}$ is a vertex, where $\rho \neq \theta$.*

Lemma 9 ([9, Lemma 2.4])**.** *In an a^4b-tiling, the following inequalities hold.*

1. *If $\delta, \epsilon < 1$, then $\alpha + 2\beta > 1$ and $\alpha + 2\gamma > 1$.*
2. *If $\beta, \delta < 1$, then $2\alpha + \gamma > 1$ and $\gamma + 2\epsilon > 1$.*
3. *If $\gamma, \epsilon < 1$, then $2\alpha + \beta > 1$ and $\beta + 2\delta > 1$.*

Lemma 10 ([13, Lemma 2])**.** *In an a^4b-tiling, $\beta > \gamma$ is equivalent to $\delta < \epsilon$, and $\beta < \gamma$ is equivalent to $\delta > \epsilon$.*

Lemma 11 ([5, Lemma 18])**.** *If the angles $\alpha, \beta, \gamma, \delta, \epsilon$ and the edge length a of an almost equilateral pentagon satisfy $a \notin \mathbb{Z}\pi$, then*

$$\begin{aligned}
&[((1-\cos\beta)\sin(\delta - \tfrac{1}{2}\alpha) - (1-\cos\gamma)\sin(\epsilon - \tfrac{1}{2}\alpha))\sin\tfrac{1}{2}(\delta-\epsilon)\\
&-(1-\cos(\beta-\gamma))\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}(\delta+\epsilon)]\cos\tfrac{1}{2}(\delta+\epsilon-\alpha) = 0, \qquad (2.3)\\
&\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}(\beta-\gamma)(\sin\tfrac{1}{2}\beta\sin\delta\cos a - \cos\tfrac{1}{2}\beta\cos\delta)\\
&+\sin\tfrac{1}{2}\gamma\sin\tfrac{1}{2}(\delta-\epsilon)\cos\tfrac{1}{2}(\delta+\epsilon-\alpha) = 0, \qquad (2.4)\\
&\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}(\beta-\gamma)(\sin\tfrac{1}{2}\gamma\sin\epsilon\cos a - \cos\tfrac{1}{2}\gamma\cos\epsilon)\\
&+\sin\tfrac{1}{2}\beta\sin\tfrac{1}{2}(\delta-\epsilon)\cos\tfrac{1}{2}(\delta+\epsilon-\alpha) = 0. \qquad (2.5)
\end{aligned}$$

Conversely, if $\alpha, \beta, \gamma, \delta, \epsilon, a$ satisfy the three equalities, and $\alpha, \beta - \gamma \notin 2\mathbb{Z}\pi$, and one of $\delta, \epsilon \notin \mathbb{Z}\pi$, then there is an almost equilateral pentagon with the given $\alpha, \beta, \gamma, \delta, \epsilon, a$.

Lemma 12. *Given α, β, γ and a, the following formula solves b:*

$$\cos b \quad = \quad B_4 \cos^4 a + B_3 \cos^3 a + B_2 \cos^2 a + B_1 \cos a + B_0 \qquad (2.6)$$

$$\begin{aligned} B_4 &= (1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma) \\ B_3 &= -B_1 = \sin\alpha\,(\sin(\beta+\gamma) - \sin\beta - \sin\gamma) \\ B_2 &= \cos\alpha\,(\cos(\beta-\gamma) - \cos\beta - \cos\gamma + 1) - 2\cos\beta\cos\gamma + \cos\beta + \cos\gamma \\ B_0 &= \cos\beta\cos\gamma - \cos\alpha\sin\beta\sin\gamma. \end{aligned}$$

The proof is similar to [12, Lemma 11].

A vertex $\alpha^{n_1}\beta^{n_2}\gamma^{n_3}\delta^{n_4}\epsilon^{n_5}$ can be effectively represented by its vector type $\boldsymbol{n} = (n_1, n_2, n_3, n_4, n_5) \in \mathbb{N}^5$. For convenience, both representations will be alternately used later.

Lemma 13 (Irrational Angle Lemma)**.** *Given three different vertices $\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m} \in \mathbb{N}^5$ in an a^4b-tiling with any irrational angle and f tiles, if $\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}$ and $\boldsymbol{u} :=$ (1 1 1 1 1) are linearly independent, then any other vertex $\boldsymbol{n}$ belongs to*

$$\mathbb{N}^5 \cap 3 \text{ dimensional affine subspace containing } \left\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, \frac{2f}{3f+4}\boldsymbol{u}\right\}.$$

In particular, the square matrix of $\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{u}, \boldsymbol{n}$ has zero determinant.

Proof. By Lemma 3 and the vertices $\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{n}$, the angles satisfy a linear system of equations with the augmented matrix

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 3+\frac{4}{f} \\ k_1 & k_2 & k_3 & k_4 & k_5 & 2 \\ l_1 & l_2 & l_3 & l_4 & l_5 & 2 \\ m_1 & m_2 & m_3 & m_4 & m_5 & 2 \\ n_1 & n_2 & n_3 & n_4 & n_5 & 2 \end{pmatrix}.$$

If $\boldsymbol{n}$ is not a linear combination of $\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{u}$, then the system has a unique rational solution, a contradiction. Therefore, the last row of C must be some linear combination of the other 4 rows, which implies that $\boldsymbol{n}$ lies in a three-dimensional affine subspace containing 4 points $\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, \frac{2f}{3f+4}\boldsymbol{u}$. □

Since there is no b^2-angle, any degree 3 vertex is either an a^3-vertex:

$$\{\alpha^3, \alpha^2\beta, \alpha^2\gamma, \alpha\beta\gamma, \alpha\beta^2, \alpha\gamma^2, \beta^3, \beta^2\gamma, \beta\gamma^2, \gamma^3\}$$

or an a^2b-vertex:

$$\{\alpha\delta^2, \alpha\delta\epsilon, \alpha\epsilon^2, \beta\delta^2, \beta\delta\epsilon, \beta\epsilon^2, \gamma\delta^2, \gamma\delta\epsilon, \gamma\epsilon^2\}.$$

Lemma 14. *A non-symmetric a^4b-tiling with general angles can have at most two different types of a^2b-vertex.*

Proof. For a^2b-vertices with no α, Lemma 5 in [13] implies that there are at most 2 different types of such vertices, and that the only possible combinations of two such vertices are $\{\beta\delta^2, \gamma\epsilon^2\}$, $\{\beta\delta\epsilon, \gamma\epsilon^2\}$, or $\{\beta\delta^2, \gamma\delta\epsilon\}$. Up to the symmetry of exchanging $\beta \leftrightarrow \gamma$, $\delta \leftrightarrow \epsilon$, it is enough to consider $\{\beta\delta^2, \gamma\epsilon^2\}$ or $\{\beta\delta\epsilon, \gamma\epsilon^2\}$.

If an a^2b-vertex has α, then it is $\alpha\delta^2$, $\alpha\delta\epsilon$ or $\alpha\epsilon^2$. These three vertices cannot appear simultaneously (otherwise $\delta = \epsilon$ and the pentagon is symmetric). We only need to show that the following six combinations of three a^2b-vertices are impossible:

1. $\alpha\delta\epsilon, \beta\delta^2, \gamma\epsilon^2$ 2. $\alpha\delta\epsilon, \beta\delta\epsilon, \gamma\epsilon^2$ 3. $\alpha\delta^2, \beta\delta^2, \gamma\epsilon^2$

4. $\alpha\delta^2, \beta\delta\epsilon, \gamma\epsilon^2$ 5. $\alpha\epsilon^2, \beta\delta^2, \gamma\epsilon^2$ 6. $\alpha\epsilon^2, \beta\delta\epsilon, \gamma\epsilon^2$

Table 4: All possible cases with three a^2b-vertex types.

Vertex	Angles	Irrational Angle Lemma	AAD	Contradiction
$\alpha\delta\epsilon, \beta\delta^2, \gamma\epsilon^2$	$(\frac{1}{2}+\frac{2}{f}, 2-2\delta, -1+2\delta+\frac{4}{f}, \delta, \frac{3}{2}-\delta-\frac{2}{f})$	$2n_2+n_5 = 2n_3+n_4$	$\lvert^\delta\beta^\alpha\rvert^\beta\delta^\epsilon\rvert^\epsilon\delta^\beta\rvert$	$\alpha\beta\cdots = \alpha\beta\gamma \Rightarrow \delta = \epsilon$
$\alpha\delta\epsilon, \beta\delta\epsilon, \gamma\epsilon^2$	$(1-\frac{2}{3}\delta+\frac{4}{3f}, 1-\frac{2}{3}\delta+\frac{4}{3f}, \frac{2}{3}\delta+\frac{8}{3f}, \delta, 1-\frac{1}{3}\delta-\frac{4}{3f})$	$2n_1+2n_2+n_5 = 2n_3+3n_4$	$\lvert^\epsilon\gamma^\alpha\rvert^\gamma\epsilon^\delta\rvert^\delta\epsilon^\gamma\rvert$	No $\alpha\gamma\cdots$ by the
$\alpha\delta^2, \beta\delta^2, \gamma\epsilon^2$	$(\frac{2}{3}\epsilon+\frac{8}{3f}, \frac{2}{3}\epsilon+\frac{8}{3f}, 2-2\epsilon, 1-\frac{1}{3}\epsilon-\frac{4}{3f}, \epsilon)$	$2n_1+2n_2+3n_5 = 6n_3+n_4$		irrational angle lemma
$\alpha\delta^2, \beta\delta\epsilon, \gamma\epsilon^2$	$(2-2\delta, \frac{1}{2}+\frac{2}{f}, -1+2\delta+\frac{4}{f}, \delta, \frac{3}{2}-\delta-\frac{2}{f})$	$2n_1+n_5 = 2n_3+n_4$	$\lvert^\gamma\alpha^\beta\rvert^\beta\delta^\epsilon\rvert^\epsilon\delta^\beta\rvert$	No $\beta^2\cdots$
$\alpha\epsilon^2, \beta\delta^2, \gamma\epsilon^2$	$(\frac{2}{3}\delta+\frac{8}{3f}, 2-2\delta, \frac{2}{3}\delta+\frac{8}{3f}, \delta, 1-\frac{1}{3}\delta-\frac{4}{3f})$	$2n_1+2n_3+3n_4 = 6n_2+n_5$	$\lvert^\beta\alpha^\gamma\rvert^\gamma\epsilon^\delta\rvert^\delta\epsilon^\gamma\rvert$	No $\gamma^2\cdots$
$\alpha\epsilon^2, \beta\delta\epsilon, \gamma\epsilon^2$	$(\frac{1}{2}+\frac{2}{f}, \frac{5}{4}-\delta+\frac{1}{f}, \frac{1}{2}+\frac{2}{f}, \delta, \frac{3}{4}-\frac{1}{f})$	$n_2 = n_4$	$\lvert^\epsilon\gamma^\alpha\rvert^\gamma\epsilon^\beta\rvert^\beta\epsilon^\gamma\rvert$	No $\delta^2\cdots$

Case 1. $\{\alpha\delta\epsilon, \beta\delta^2, \gamma\epsilon^2\}$

We will explain the first row of Table 4. The AAD of $\lvert^\delta\beta^\alpha\rvert^\beta\delta^\epsilon\rvert^\epsilon\delta^\beta\rvert$ gives $\alpha\beta\cdots$, denoted by $\boldsymbol{n}$ with $n_1 \geq 1, n_2 \geq 1$. By the irrational angle lemma, we have $2n_2 + n_5 = 2n_3 + n_4 \geq 2$. Let $R(\alpha\beta\cdots)$ denote the remainder or "$\cdots$" part of the vertex $\alpha\beta\cdots$. By the angle expressions in Table 4, we get $R(\alpha\beta\cdots) < 2\delta, \delta+\epsilon, 2\epsilon, 2\gamma$. By the parity lemma, we must have $n_4 = n_5 = 0$. By the parity lemma, we have $n_3 = 1$, i.e. $\alpha\beta\cdots = \alpha\beta\gamma$. This new vertex implies $f = 12$. By the angle equation (2.3) in Lemma 11, we get $\delta = \epsilon = \frac{2}{3}$, a contradiction.

The other five cases also lead to contradictions similarly following Table 4. □

In fact, the above proof also induces the following two lemmas.

Lemma 15. *In a non-symmetric a^4b-tiling, if there are only two a^2b-vertex types, then it must be one of the following (up to the exchange $\beta \leftrightarrow \gamma$, $\delta \leftrightarrow \epsilon$):*

1. $\beta\delta^2, \gamma\epsilon^2$ 2. $\beta\delta\epsilon, \gamma\epsilon^2$ 3. $\alpha\delta^2, \beta\delta^2$ 4. $\alpha\delta^2, \beta\delta\epsilon$ 5. $\alpha\delta^2, \beta\epsilon^2$ 6. $\alpha\delta^2, \gamma\delta^2$

7. $\alpha\delta^2, \gamma\delta\epsilon$ 8. $\alpha\delta^2, \gamma\epsilon^2$ 9. $\alpha\delta\epsilon, \beta\delta^2$ 10. $\alpha\delta\epsilon, \beta\epsilon^2$ 11. $\alpha\delta\epsilon, \beta\delta\epsilon$

Lemma 16. *In a non-symmetric a^4b-tiling, if there is only one a^2b-vertex type, then it must be one of the following (up to the exchange $\beta \leftrightarrow \gamma$, $\delta \leftrightarrow \epsilon$):*

1. $\alpha\delta^2$ 2. $\beta\delta^2$ 3. $\gamma\delta^2$ 4. $\alpha\delta\epsilon$ 5. $\beta\delta\epsilon$

3 Tilings with two a^2b-vertex types

To facilitate discussion, we denote by T_i the tile labeled i, by θ_i the angle θ in T_i. When we say a tile is *determined*, we mean that we know all the edges and angles of the tile.

For the eleven pairs of a^2b-vertices in Lemma 15, this section will show that only $\{\alpha\delta\epsilon, \beta\delta\epsilon\}$ admits tilings.

Proposition 1. *There is no non-symmetric a^4b-tiling with general angles, such that both members of any one of the first ten pairs in Lemma 15 are vertices.*

Proof. For each pair of a^2b-vertices, we will first apply Lemma 8 to get a new vertex, then apply the irrational angle lemma to control the AVC, and finally try to get a contradiction by the AAD. We summarize such computations and deductions in Table 5 and 6.

For example, when $\{\alpha\delta^2, \gamma\epsilon^2\}$ are vertices, if β does not appear in degree 3 vertices, by Lemma 8, one of $\{\alpha\beta^3, \beta^3\gamma, \beta^4, \beta^5\}$ must appear. If β appears in a degree 3 vertex, by Lemma 14, this vertex must be an a^3-vertex i.e. one of $\{\alpha^2\beta, \alpha\beta^2, \alpha\beta\gamma, \beta^3, \beta^2\gamma, \beta\gamma^2\}$. Applying the irrational angle lemma to each case, we get the first ten rows of Table 6 for the pair $\{\alpha\delta^2, \gamma\epsilon^2\}$. We show the case $\{\alpha\delta^2, \gamma\epsilon^2, \beta\gamma^2\}$ in detail.

By the irrational angle lemma, we have $6n_1 + 2n_3 = 4n_2 + 3n_4 + n_5$. The angles are

$$\alpha = 2 - 2\delta,\ \beta = -\frac{2}{3} + \frac{4\delta}{3} + \frac{16}{3f},\ \gamma = \frac{4}{3} - \frac{2\delta}{3} - \frac{8}{3f},\ \epsilon = \frac{1}{3} + \frac{\delta}{3} + \frac{4}{3f}$$

The AAD of $|^{\gamma}\alpha^{\beta}|^{\beta}\delta^{\epsilon}|^{\epsilon}\delta^{\beta}|$ gives $\beta^2\cdots$, denoted by $\boldsymbol{n}$ with $n_2 \geq 2$. By $R(\beta^2\cdots) < \alpha+\gamma, 4\gamma$ and $6n_1+2n_3 = 4n_2+3n_4+n_5 \geq 8$, we get $n_1 \geq 2, n_3 = 0$. Then $4n_2+3n_4+n_5 = 6n_1 \geq 12$. So we get $(n_2-2)+n_4+n_5 \geq 1$, contradicting

$$2\alpha + 3\beta > 2, 2\alpha + 2\beta + \delta > 2 \text{ and } 2\alpha + 2\beta + \epsilon > 2.$$

Therefore, Case $\{\alpha\delta^2, \gamma\epsilon^2, \beta\gamma^2\}$ admits no tiling.

When the new vertex is $\alpha\beta\gamma$, β^3, β^4 or β^5, we actually deduce a combinatorial tiling of the sphere. However, there is no solution to (2.3), (2.4), (2.5) in Lemma 11 (see also [5, Proposition 20]), i.e. there are no such pentagons geometrically.

All other pairs are discussed similarly by Table 5 and 6. □

Proposition 2. *All non-symmetric a^4b-tilings with general angles, such that $\alpha\delta\epsilon$, $\beta\delta\epsilon$ appear as vertices, are the following:*

- *A unique tiling $T(8\alpha\gamma^2, 8\alpha\delta\epsilon, 8\beta\delta\epsilon, 2\beta^4)$ (see the second picture of Figure 7) with $f = 16$, by the unique pentagon with angles $(\frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{3}{2} - \epsilon, \epsilon)$, $\cos(\epsilon\pi) = 1/\sqrt[4]{2}$, and with sides $\cos a = \sqrt{\sqrt{2}-1}$, $\cos b = \sqrt{2}\sqrt{\sqrt{2}-1}$.*

Table 5: All possible cases with two a^2b-vertex types, part one.

Vertex		Angles	Irrational Angle Lemma	AAD	Contradiction
$\beta\delta^2, \gamma\epsilon^2$	$\alpha^2\beta$	$(\delta, 2-2\delta, \frac{3}{f}, \delta, 1-\frac{4}{f})$	$n_1+n_4=2n_2$	$\lvert^{\epsilon}\gamma^{\alpha}\rvert\gamma\epsilon^{\delta}\rvert^{\delta}\epsilon^{\gamma}\rvert$	No $\alpha\gamma\cdots$ by the irrational angle lemma
	$\alpha\beta^2$	$(-2+4\delta, 2-2\delta, 4-6\delta+\frac{8}{f}, \delta, -1+3\delta-\frac{4}{f})$	$4n_1+n_4+3n_5=2n_2+6n_3$		
	$\alpha^3\beta$	$(\frac{2}{3}\delta, 2-2\delta, \frac{2}{3}\delta+\frac{8}{f}, \delta, 1-\frac{1}{3}\delta-\frac{4}{f})$	$2n_1+2n_3+3n_4=6n_2+n_5$		
	$\alpha^2\gamma$	$(\epsilon, \frac{8}{f}, 2-2\epsilon, 1-\frac{4}{f}, \epsilon)$	$n_1+n_5=2n_3$	$\lvert^{\delta}\beta^{\alpha}\rvert^{\beta}\delta^{\epsilon}\rvert^{\epsilon}\delta^{\beta}\rvert$	No $\alpha\beta\cdots$
	$\alpha\gamma^2$	$(-\frac{2}{3}+\frac{4}{3}\delta+\frac{10}{3f}, 2-2\delta, \frac{4}{3}-\frac{2}{3}\delta-\frac{8}{3f}, \delta, \frac{1}{3}+\frac{1}{3}\delta+\frac{4}{3f})$	$4n_1+3n_4+n_5=6n_2+2n_3$		
	$\alpha^3\gamma$	$(2-2\delta-\frac{8}{f}, 2-2\delta, -4+6\delta+\frac{24}{f}, \delta, 3-3\delta-\frac{12}{f})$	$2n_1+2n_2+3n_5=6n_3+n_4$		
	$\alpha\beta\gamma$	$(\frac{8}{f}, 2-2\delta, 2\delta-\frac{8}{f}, \delta, 1-\delta+\frac{4}{f})$	$2n_2+n_5=2n_3+n_4$		No solution to (2.3), (2.4), (2.5) in Lemma 11
	α^3	$(\frac{2}{3}, 2-2\delta, -\frac{4}{3}+2\delta+\frac{8}{f}, \delta, \frac{5}{3}-\delta-\frac{4}{f})$			
	α^4	$(\frac{1}{2}, 2-2\delta, -1+2\delta+\frac{8}{f}, \delta, \frac{3}{2}-\delta-\frac{4}{f})$			
	α^5	$(\frac{2}{5}, 2-2\delta, -\frac{4}{5}+2\delta+\frac{8}{f}, \delta, \frac{7}{5}-\delta-\frac{4}{f})$			
$\beta\delta\epsilon, \gamma\epsilon^2$	$\alpha\beta^2$	$(-1+2\epsilon+\frac{4}{f}, \frac{3}{2}-\epsilon-\frac{2}{f}, 2-2\epsilon, \frac{1}{2}+\frac{2}{f}, \epsilon)$	$2n_1+n_5=n_2+2n_3$	$\lvert^{\epsilon}\gamma^{\alpha}\rvert\gamma\epsilon^{\delta}\rvert^{\delta}\epsilon^{\gamma}\rvert$	No $\alpha\gamma\cdots$
	$\alpha^2\beta$	$(\frac{1}{3}+\frac{2\delta}{3}-\frac{4}{3f}, \frac{4}{3}-\frac{4}{3}\delta+\frac{8}{3f}, \frac{2}{3}-\frac{8}{3}\delta+\frac{16}{3f}, \delta, \frac{2}{3}+\frac{8}{3}\delta-\frac{8}{3f})$	$2n_1+3n_4+n_5=4n_2+2n_3$		No $\delta^2\cdots$
	$\alpha\beta\gamma$	$(1-2\delta+\frac{12}{f}, 1-\frac{4}{f}, 2\delta-\frac{8}{f}, \delta, 1-\delta+\frac{4}{f})$	$2n_1+n_5=2n_3+n_4$		
	$\alpha^3\beta$	$(\frac{1}{5}+\frac{2}{5}\delta-\frac{4}{5f}, \frac{7}{5}-\frac{6}{5}\delta+\frac{12}{5f}, \frac{4}{5}-\frac{2}{5}\delta+\frac{24}{5f}, \delta, \frac{3}{5}+\frac{1}{5}\delta-\frac{12}{5f})$	$2n_1+5n_4+n_5=6n_2+2n_3$		
	$\alpha^2\gamma$	$(1-\frac{4}{f}, 1-\delta+\frac{4}{f}, \frac{8}{f}, \delta, 1-\frac{4}{f})$	$n_2=n_4$		
	$\alpha\gamma^2$	$(\frac{8}{f}, \frac{3}{2}-\delta-\frac{2}{f}, 1-\frac{4}{f}, \delta, \frac{1}{2}+\frac{2}{f})$			
	$\alpha^3\gamma$	$(\frac{1}{2}-\frac{2}{f}, \frac{3}{4}-\delta+\frac{3}{f}, \frac{1}{2}+\frac{6}{f}, \delta, \frac{3}{4}-\frac{3}{f})$			
	α^3	$(\frac{2}{3}, \frac{7}{6}-\delta+\frac{2}{f}, \frac{1}{3}+\frac{4}{f}, \delta, \frac{5}{6}-\frac{2}{f})$			
	α^4	$(\frac{1}{2}, \frac{5}{4}-\delta+\frac{2}{f}, \frac{1}{2}+\frac{4}{f}, \delta, \frac{3}{4}-\frac{2}{f})$			
	α^5	$(\frac{2}{5}, \frac{13}{10}-\delta+\frac{2}{f}, \frac{3}{5}+\frac{4}{f}, \delta, \frac{7}{10}-\frac{2}{f})$			
$\alpha\delta^2, \beta\delta^2$	$\alpha^2\gamma$	$(2-2\delta, 2-2\delta, -2+4\delta, \delta, 1-\delta+\frac{4}{f})$	$2n_1+2n_2+n_5=4n_3+n_4$	$\lvert\gamma\alpha^{\beta}\rvert\beta\delta^{\epsilon}\rvert^{\epsilon}\delta^{\beta}\rvert$	No $\epsilon^2\cdots$
	$\alpha\beta\gamma$				
	$\beta^2\gamma$				
	$\alpha\gamma^2$	$(2-2\delta, 2-2\delta, \delta, \delta, -1+2\delta+\frac{4}{f})$	$2n_1+2n_2=n_3+n_4+2n_5$		
	$\beta\gamma^2$				
	$\alpha\gamma^3$	$(2-2\delta, 2-2\delta, \frac{2\delta}{3}, \delta, -1+\frac{7\delta}{3}+\frac{4}{f})$	$6n_1+6n_2=2n_3+3n_4+7n_5$		
	$\beta\gamma^3$				
	γ^3	$(2-2\delta, 2-2\delta, \frac{2}{3}, \delta, -\frac{5}{3}+3\delta+\frac{4}{f})$	$2n_1+2n_2=n_4+3n_5$		
	γ^4	$(2-2\delta, 2-2\delta, \frac{1}{2}, \delta, -\frac{3}{2}+3\delta+\frac{4}{f})$			
	γ^5	$(2-2\delta, 2-2\delta, \frac{2}{5}, \delta, -\frac{7}{5}+3\delta+\frac{4}{f})$			
$\alpha\delta^2, \beta\delta\epsilon$	$\alpha\beta\gamma$	$(2-2\delta, 1-\frac{4}{f}, -1+2\delta+\frac{4}{f}, \delta, 1-\delta+\frac{4}{f})$	$2n_1+n_5=2n_3+n_4$	$\lvert\gamma\alpha^{\beta}\rvert\beta\delta^{\epsilon}\rvert^{\epsilon}\delta^{\beta}\rvert$	No $\epsilon^2\cdots$
	$\beta^2\gamma$	$(2-2\delta, \frac{3}{2}-\delta-\frac{2}{f}, -1+2\delta+\frac{4}{f}, \delta, \frac{1}{2}+\frac{2}{f})$	$2n_1+n_2=2n_3+n_4$		
	$\beta\gamma^2$	$(2-2\delta, 4-4\delta-\frac{8}{f}, -1+2\delta+\frac{4}{f}, \delta, -2+3\delta+\frac{8}{f})$	$2n_1+4n_2=2n_3+n_4+3n_5$		
	$\beta\gamma^3$	$(2-2\delta, 5-6\delta-\frac{12}{f}, -1+2\delta+\frac{4}{f}, \delta, -3+5\delta+\frac{12}{f})$	$2n_1+6n_2=2n_3+n_4+5n_5$		
	$\alpha^2\gamma$	$(1-\frac{4}{f}, \frac{3}{2}-\epsilon-\frac{2}{f}, \frac{8}{f}, \frac{1}{2}+\frac{2}{f}, \epsilon)$	$n_2=n_5$		
	$\alpha\gamma^2$	$(\frac{8}{f}, 1-\epsilon+\frac{4}{f}, 1-\frac{4}{f}, 1-\frac{4}{f}, \epsilon)$			
	$\alpha\gamma^3$	$(\frac{1}{2}+\frac{6}{f}, \frac{5}{4}-\epsilon+\frac{3}{f}, \frac{1}{2}-\frac{2}{f}, \frac{3}{4}-\frac{3}{f}, \epsilon)$			
	γ^3	$(\frac{1}{3}+\frac{4}{f}, \frac{7}{6}-\epsilon+\frac{2}{f}, \frac{1}{3}, \frac{5}{6}-\frac{2}{f}, \epsilon)$			
	γ^4	$(\frac{1}{2}+\frac{4}{f}, \frac{5}{4}-\epsilon+\frac{2}{f}, \frac{1}{2}, \frac{3}{4}-\frac{2}{f}, \epsilon)$			
	γ^5	$(\frac{3}{5}+\frac{4}{f}, \frac{13}{10}-\epsilon+\frac{2}{f}, \frac{2}{5}, \frac{7}{10}-\frac{2}{f}, \epsilon)$			
$\alpha\delta^2, \beta\epsilon^2$	$\alpha\gamma^2$	$(2-2\delta, \frac{2}{f}, \delta, \delta, 1-\frac{4}{f})$	$2n_1=n_3+n_4$	$\lvert\alpha\beta^{\delta}\rvert\gamma\epsilon^{\delta}\rvert^{\epsilon}\gamma\rvert$	No $\gamma\delta\cdots$
	$\beta^2\gamma$	$(2-2\delta, \frac{4}{3}-\frac{2}{3}\delta-\frac{8}{3f}, -\frac{2}{3}+\frac{4}{3}\delta+\frac{16}{3f}, \delta, \frac{1}{3}+\frac{1}{3}\delta+\frac{4}{3f})$	$6n_1+2n_2=4n_3+3n_4+n_5$		
	γ^3	$(2-2\delta, -\frac{4}{3}+2\delta+\frac{8}{f}, \frac{2}{3}, \delta, \frac{5}{3}-\delta-\frac{4}{f})$	$2n_1+n_5=2n_2+n_4$	$\lvert^{\epsilon}\gamma^{\alpha}\rvert^{\epsilon}\gamma^{\alpha}\rvert\cdots$	No $\alpha\epsilon\cdots$
	$\alpha\beta\gamma$	$(2-2\delta, 2\delta-\frac{8}{f}, \frac{8}{f}, \delta, 1-\delta+\frac{4}{f})$			
	$\alpha^2\gamma$	$(2-2\delta, 4-6\delta+\frac{8}{f}, -2+4\delta, \delta, -1+3\delta-\frac{4}{f})$	$2n_1+6n_2=4n_3+n_4+3n_5$		No $\beta^2\cdots$
	$\beta\gamma^2$	$(\frac{8}{f}, 2-2\epsilon, \epsilon, 1-\frac{4}{f}, \epsilon)$	$2n_2=n_3+n_5$		
	$\alpha\gamma^3$	$(2-2\delta, \frac{2}{3}\delta+\frac{8}{f}, \frac{2}{3}\delta, \delta, 1-\frac{1}{3}\delta-\frac{4}{f})$	$6n_1+n_5=2n_2+2n_3+3n_4$		
	$\beta\gamma^3$	$(2-2\delta, -4+6\delta+\frac{24}{f}, 2-2\delta-\frac{8}{f}, \delta, 3-3\delta-\frac{12}{f})$	$2n_1+2n_3+3n_5=6n_2+n_4$		
	γ^4	$(2-2\delta, -1+2\delta+\frac{8}{f}, \frac{1}{2}, \delta, \frac{3}{2}-\delta-\frac{4}{f})$	$2n_1+n_5=2n_2+n_4$		
	γ^5	$(2-2\delta, -\frac{4}{5}+2\delta+\frac{8}{f}, \frac{2}{5}, \delta, \frac{7}{5}-\delta-\frac{4}{f})$			
$\alpha\delta^2, \gamma\delta^2$	$\alpha^2\beta$	$(2-2\delta, -2+4\delta, 2-2\delta, \delta, 1-\delta+\frac{4}{f})$	$2n_1+2n_3+n_5=4n_2+n_4$	$\lvert\gamma\alpha^{\beta}\rvert\beta\delta^{\epsilon}\rvert^{\epsilon}\delta^{\beta}\rvert$	No $\epsilon^2\cdots$
	$\alpha\beta\gamma$				
	$\beta\gamma^2$				
	$\alpha\beta^2$	$(2-2\delta, \delta, 2-2\delta, \delta, -1+2\delta+\frac{4}{f})$	$2n_1+2n_3=n_2+n_4+2n_5$		
	$\beta^2\gamma$				
	$\alpha\beta^3$	$(2-2\delta, \frac{2}{3}\delta, 2-2\delta, \delta, -1+\frac{7}{3}\delta+\frac{4}{f})$	$6n_1+6n_3=2n_2+3n_4+7n_5$		
	$\beta^3\gamma$				
	β^3	$(2-2\delta, \frac{2}{3}, 2-2\delta, \delta, -\frac{5}{3}+3\delta+\frac{4}{f})$	$2n_1+2n_3=n_4+3n_5$		
	β^4	$(2-2\delta, \frac{1}{2}, 2-2\delta, \delta, -\frac{3}{2}+3\delta+\frac{4}{f})$			
	β^5	$(2-2\delta, \frac{2}{5}, 2-2\delta, \delta, -\frac{7}{5}+3\delta+\frac{4}{f})$			
$\alpha\delta^2, \gamma\delta\epsilon$	$\alpha\beta\gamma$	$(2-2\delta, -1+2\delta+\frac{4}{f}, 1-\frac{4}{f}, \delta, 1-\delta+\frac{4}{f})$	$2n_1+n_5=2n_2+n_4$		No $\beta^2\cdots$
	$\beta\gamma^2$	$(2-2\delta, -1+2\delta+\frac{4}{f}, \frac{3}{2}-\delta-\frac{2}{f}, \delta, \frac{1}{2}+\frac{2}{f})$	$2n_1+n_3=2n_2+n_4$		
	$\beta^2\gamma$	$(2-2\delta, -1+2\delta+\frac{4}{f}, 4-4\delta-\frac{8}{f}, \delta, -2+3\delta+\frac{8}{f})$	$2n_1+4n_3=2n_2+n_4+3n_5$		No $\epsilon^2\cdots$
	$\beta^3\gamma$	$(2-2\delta, -1+2\delta+\frac{4}{f}, 5-6\delta-\frac{12}{f}, \delta, -3+5\delta+\frac{12}{f})$	$2n_1+6n_3=2n_2+n_4+5n_5$		
	$\alpha^2\beta$	$(1-\frac{4}{f}, \frac{8}{f}, \frac{3}{2}-\epsilon-\frac{2}{f}, \frac{1}{2}+\frac{2}{f}, \epsilon)$	$n_3=n_5$		
	$\alpha\beta^2$	$(\frac{8}{f}, 1-\frac{4}{f}, 1-\epsilon+\frac{4}{f}, 1-\frac{4}{f}, \epsilon)$			
	$\alpha\beta^3$	$(\frac{1}{2}+\frac{6}{f}, \frac{1}{2}-\frac{2}{f}, \frac{5}{4}-\epsilon+\frac{3}{f}, \frac{3}{4}-\frac{3}{f}, \epsilon)$			
	β^3	$(\frac{1}{3}+\frac{4}{f}, \frac{2}{3}, \frac{7}{6}-\epsilon+\frac{2}{f}, \frac{5}{6}-\frac{2}{f}, \epsilon)$			
	β^4	$(\frac{1}{2}+\frac{4}{f}, \frac{1}{2}, \frac{5}{4}-\epsilon+\frac{2}{f}, \frac{3}{4}-\frac{2}{f}, \epsilon)$			
	β^5	$(\frac{3}{5}+\frac{4}{f}, \frac{2}{5}, \frac{13}{10}-\epsilon+\frac{2}{f}, \frac{7}{10}-\frac{2}{f}, \epsilon)$			

Table 6: All possible cases with two a^2b-vertex types, part two.

Vertex		Angles	Irrational Angle Lemma	AAD	Contradiction
$\alpha\delta^2, \gamma\epsilon^2$	$\alpha^2\beta$	$(2-2\delta, -2+4\delta, 4-6\delta+\frac{8}{f}, \delta, -1+3\delta-\frac{4}{f})$	$2n_1+6n_3=4n_2+n_4+3n_5$	$\lvert^\gamma\alpha^\beta\rvert^\beta\delta^\epsilon\rvert^\epsilon\delta^\beta\rvert$	No $\beta^2\cdots$
	$\beta\gamma^2$	$(2-2\delta, -\frac{2}{3}+\frac{4\delta}{3}+\frac{16}{3f}, \frac{4}{3}-\frac{2\delta}{3}-\frac{8}{3f}, \delta, \frac{1}{3}+\frac{\delta}{3}+\frac{4}{3f})$	$6n_1+2n_3=4n_2+3n_4+n_5$		
	$\alpha\beta^2$	$(2-2\delta, \delta, \frac{8}{f}, \delta, 1-\frac{4}{f})$	$2n_1=n_2+n_4$	$\lvert^\epsilon\gamma^\alpha\rvert^\gamma\epsilon^\delta\rvert^\delta\epsilon^\gamma\rvert$	No $\alpha\gamma\cdots$
	$\beta^2\gamma$	$(\frac{8}{f}, \epsilon, 2-2\epsilon, 1-\frac{4}{f}, \epsilon)$	$n_2+n_5=2n_3$		
	$\alpha\beta^3$	$(2-2\delta, \frac{2\delta}{3}, \frac{2\delta}{3}+\frac{8}{f}, \delta, 1-\frac{\delta}{3}-\frac{4}{f})$	$6n_1+n_5=2n_2+2n_3+3n_4$		
	$\beta^3\gamma$	$(2-2\delta, 2-2\delta-\frac{8}{f}, -4+6\delta+\frac{24}{f}, \delta, 3-3\delta-\frac{12}{f})$	$2n_1+2n_2+3n_5=6n_3+n_4$		
	$\alpha\beta\gamma$	$(2-2\delta, \frac{8}{f}, 2\delta-\frac{8}{f}, \delta, 1-\delta+\frac{4}{f})$	$2n_1+n_5=2n_3+n_4$		No solution to (2.3), (2.4), (2.5) in Lemma 11
	β^3	$(2-2\delta, \frac{2}{3}, -\frac{4}{3}+2\delta+\frac{8}{f}, \delta, \frac{5}{3}-\delta-\frac{4}{f})$			
	β^4	$(2-2\delta, \frac{1}{2}, -1+2\delta+\frac{8}{f}, \delta, \frac{3}{2}-\delta-\frac{4}{f})$			
	β^5	$(2-2\delta, \frac{2}{5}, -\frac{4}{5}+2\delta+\frac{8}{f}, \delta, \frac{7}{5}-\delta-\frac{4}{f})$			
$\alpha\delta\epsilon, \beta\delta^2$	$\alpha^2\gamma$	$(\frac{3}{2}-\delta-\frac{2}{f}, 2-2\delta, -1+2\delta+\frac{4}{f}, \delta, \frac{1}{2}+\frac{2}{f})$	$n_1+2n_2=2n_3+n_4$	$\lvert^\delta\beta^\alpha\rvert^\beta\delta^\epsilon\rvert^\epsilon\delta^\beta\rvert$	No $\alpha\beta\cdots$
	$\alpha\beta\gamma$	$(1-\frac{4}{f}, 2-2\delta, -1+2\delta+\frac{4}{f}, \delta, 1-\delta+\frac{4}{f})$	$2n_2+n_5=2n_3+n_4$		No $\epsilon^2\cdots$
	$\alpha\gamma^2$	$(4-4\delta-\frac{8}{f}, 2-2\delta, -1+2\delta+\frac{4}{f}, \delta, -2+3\delta+\frac{8}{f})$	$4n_1+2n_2=2n_3+n_4+3n_5$		
	$\alpha\gamma^3$	$(5-6\delta-\frac{12}{f}, 2-2\delta, -1+2\delta+\frac{4}{f}, \delta, -3+5\delta+\frac{12}{f})$	$6n_1+2n_2=2n_3+n_4+5n_5$		
	$\beta^2\gamma$	$(\frac{3}{2}-\epsilon-\frac{2}{f}, 1-\frac{4}{f}, \frac{8}{f}, \frac{1}{2}+\frac{2}{f}, \epsilon)$	$n_1=n_5$		
	$\beta\gamma^2$	$(1-\epsilon+\frac{4}{f}, \frac{8}{f}, 1-\frac{4}{f}, 1-\frac{4}{f}, \epsilon)$			
	$\beta\gamma^3$	$(\frac{5}{4}-\epsilon+\frac{3}{f}, \frac{1}{2}+\frac{6}{f}, \frac{1}{2}-\frac{2}{f}, \frac{3}{4}-\frac{3}{f}, \epsilon)$			
	γ^3	$(\frac{7}{6}-\epsilon+\frac{2}{f}, \frac{1}{3}+\frac{4}{f}, \frac{2}{3}, \frac{5}{6}-\frac{2}{f}, \epsilon)$			
	γ^4	$(\frac{3}{4}-\epsilon+\frac{2}{f}, \frac{1}{2}+\frac{4}{f}, \frac{1}{2}, \frac{3}{4}-\frac{2}{f}, \epsilon)$			
	γ^5	$(\frac{13}{10}-\epsilon+\frac{2}{f}, \frac{3}{5}+\frac{4}{f}, \frac{2}{5}, \frac{7}{10}-\frac{2}{f}, \epsilon)$			
$\alpha\delta\epsilon, \beta\epsilon^2$	$\alpha^2\gamma$	$(\frac{3}{2}-\epsilon-\frac{2}{f}, 2-2\epsilon, -1+2\epsilon+\frac{4}{f}, \frac{1}{2}+\frac{2}{f}, \epsilon)$	$n_1+2n_2=2n_3+n_5$	$\lvert^\alpha\beta^\delta\rvert^\gamma\epsilon^\delta\rvert^\delta\epsilon^\gamma\rvert$	No $\gamma\delta\cdots$
	$\alpha\beta\gamma$	$(1-\frac{4}{f}, 2\delta-\frac{8}{f}, 1-2\delta+\frac{12}{f}, \delta, 1-\delta+\frac{4}{f})$	$2n_2+n_4=2n_3+n_5$		No $\delta^2\cdots$
	$\alpha\gamma^2$	$(\frac{4}{3}-\frac{4}{3}\delta+\frac{8}{3f}, \frac{2}{3}-\frac{2}{3}\delta+\frac{16}{3f}, \frac{1}{3}+\frac{2}{3}\delta-\frac{4}{3f}, \delta, \frac{2}{3}+\frac{1}{3}\delta-\frac{8}{3f})$	$4n_1+2n_2=2n_3+3n_4+n_5$		
	$\alpha\gamma^3$	$(\frac{1}{5}-\frac{2}{5}\delta+\frac{12}{5f}, \frac{1}{5}-\frac{2}{5}\delta+\frac{24}{5f}, \frac{1}{5}+\frac{2}{5}\delta-\frac{4}{5f}, \delta, \frac{2}{5}+\frac{1}{5}\delta-\frac{12}{5f})$	$6n_1+2n_2=2n_3+5n_4+n_5$		
	$\beta^2\gamma$	$(\frac{3}{2}-\delta-\frac{2}{f}, 1-\frac{4}{f}, \frac{8}{f}, \delta, \frac{1}{2}+\frac{2}{f})$	$n_1=n_4$		
	$\beta\gamma^2$	$(1-\delta+\frac{4}{f}, \frac{8}{f}, 1-\frac{4}{f}, \delta, 1-\frac{4}{f})$			
	$\beta\gamma^3$	$(\frac{5}{4}-\delta+\frac{3}{f}, \frac{1}{2}+\frac{6}{f}, \frac{1}{2}-\frac{2}{f}, \delta, \frac{3}{4}-\frac{3}{f})$			
	γ^3	$(\frac{7}{6}-\delta+\frac{2}{f}, \frac{1}{3}+\frac{4}{f}, \frac{2}{3}, \delta, \frac{5}{6}-\frac{2}{f})$			
	γ^4	$(\frac{3}{4}-\delta+\frac{2}{f}, \frac{1}{2}+\frac{4}{f}, \frac{1}{2}, \delta, \frac{3}{4}-\frac{2}{f})$			
	γ^5	$(\frac{13}{10}-\delta+\frac{2}{f}, \frac{3}{5}+\frac{4}{f}, \frac{2}{5}, \delta, \frac{7}{10}-\frac{2}{f})$			

- *Many different tilings (see Table 3 and Figure 9) with $f=8k+4$ for each $k\geq 2$, by the unique pentagon with $\alpha=\beta=1-\frac{4}{f}$ in Table 2.*

Proof. If γ does not appear at degree 3 vertices, by Lemma 8, one of $\{\alpha\gamma^3, \beta\gamma^3, \gamma^4, \gamma^5\}$ must appear. If γ appears in some a^3-vertex, one of $\{\alpha^2\gamma, \alpha\beta\gamma, \alpha\gamma^2, \beta^2\gamma, \beta\gamma^2, \gamma^3\}$ must appear. In all cases, the irrational angle lemma implies $n_4=n_5$ and we get Table 7. We discuss each case as follows.

Table 7: All cases when $\alpha\delta\epsilon$ and $\beta\delta\epsilon$ appear.

Vertex		Angles	Irrational Angle Lemma	Conclusion
$\alpha\delta\epsilon, \beta\delta\epsilon$	$\alpha^2\gamma$	$(1-\frac{4}{f}, 1-\frac{4}{f}, \frac{8}{f}, \delta, 1-\delta+\frac{4}{f})$	$n_4=n_5$	See page 18
	$\alpha\beta\gamma$			
	$\beta^2\gamma$			
	$\alpha\gamma^2$	$(\frac{8}{f}, \frac{8}{f}, 1-\frac{4}{f}, \delta, 2-\delta-\frac{8}{f})$		See page 16
	$\beta\gamma^2$			
	γ^3	$(\frac{1}{3}+\frac{4}{f}, \frac{1}{3}+\frac{4}{f}, \frac{2}{3}, \delta, \frac{5}{3}-\delta-\frac{4}{f})$		See Table 9
	$\alpha\gamma^3$	$(\frac{1}{2}+\frac{6}{f}, \frac{1}{2}+\frac{6}{f}, \frac{1}{2}-\frac{2}{f}, \delta, \frac{3}{2}-\delta-\frac{6}{f})$		
	$\beta\gamma^3$			
	γ^4	$(\frac{1}{2}+\frac{4}{f}, \frac{1}{2}+\frac{4}{f}, \frac{1}{2}, \delta, \frac{3}{2}-\delta-\frac{4}{f})$		
	γ^5	$(\frac{3}{5}+\frac{4}{f}, \frac{3}{5}+\frac{4}{f}, \frac{2}{5}, \delta, \frac{7}{5}-\delta-\frac{4}{f})$		See page 16

Case. $\{\alpha\delta\epsilon, \beta\delta\epsilon, \gamma^5\}$

If $\alpha^{n_1}\beta^{n_2}\gamma^{n_3}\delta^{n_4}\epsilon^{n_5}$ is a vertex, then we have

$$(\tfrac{3}{5}+\tfrac{4}{f})n_1+(\tfrac{3}{5}+\tfrac{4}{f})n_2+\tfrac{2}{5}n_3+(\tfrac{7}{5}-\tfrac{4}{f})n_4=2.$$

Since γ^5 is a vertex, we get $f>12$. We also have $\alpha=\beta>\frac{3}{5}, \gamma=\frac{2}{5}, \delta+\epsilon>\frac{16}{15}$. This implies $n_1\le 3, n_2\le 3, n_3\le 5, n_4=n_5\le 1$. We substitute the finitely many combinations of exponents satisfying the bounds into the equation above and solve for f. By the angle values and the parity lemma, we get all possible AVC in Table 8. Its first row "f = all" means that the angle combinations can be vertices for any f; all other rows are mutually exclusive. All possible tilings based on the AVC of Table 8 are deduced as follows. Note that the AVC $\subset\{\alpha\delta\epsilon, \beta\delta\epsilon, \gamma^5\}$ in the first row admits no solution satisfying the balance lemma.

Table 8: AVC for Case $\{\alpha\delta\epsilon, \beta\delta\epsilon, \gamma^5\}$.

f	vertex
all	$\alpha\delta\epsilon, \beta\delta\epsilon, \gamma^5$
20	$\alpha^2\gamma$, $\alpha\beta\gamma$, $\beta^2\gamma$, $\alpha\gamma^3$, $\beta\gamma^3$, $\gamma^2\delta\epsilon$
60	α^3, $\alpha^2\beta$, $\alpha\beta^2$, β^3

Table 8, $f=20$.

We have the unique AAD $\gamma^5=|^\alpha\gamma^\epsilon|^\alpha\gamma^\epsilon|^\alpha\gamma^\epsilon|\cdots$. By $\alpha\epsilon\cdots=\alpha\delta\epsilon$ and $\beta^2\cdots=\beta^2\gamma$, we can determine 16 tiles in Figure 12. This tiling is the non-symmetric 3-layer earth map tiling $T(20\alpha\delta\epsilon, 10\beta^2\gamma, 2\gamma^5)$. But there is no $\beta\delta\epsilon$ in this tiling, so it actually belongs to Section 6.

Table 8, $f=60$.

We have the unique AAD for $\beta\delta\epsilon=|\beta|^\gamma\epsilon^\delta|^\epsilon\delta^\beta|$. This gives a vertex $\alpha\gamma\cdots$ or $\gamma\delta\cdots$, contradicting the AVC.

In summary, Case $\{\alpha\delta\epsilon, \beta\delta\epsilon, \gamma^5\}$ admits no tiling.

Cases $\{\alpha\delta\epsilon, \beta\delta\epsilon, \gamma^3\}$, $\{\alpha\delta\epsilon, \beta\delta\epsilon, \gamma^4\}$ and $\{\alpha\delta\epsilon, \beta\delta\epsilon, \alpha\gamma^3(\beta\gamma^3)\}$ are similar to Case $\{\alpha\delta\epsilon, \beta\delta\epsilon, \gamma^5\}$ and admit no tiling. We only give the AVC in Table 9, and skip the details.

Case. $\{\alpha\delta\epsilon, \beta\delta\epsilon, \alpha\gamma^2(\beta\gamma^2)\}$

The angle sums of $\alpha\delta\epsilon, \beta\delta\epsilon, \alpha\gamma^2(\beta\gamma^2)$ imply $\alpha=\beta=\frac{8}{f}, \gamma=1-\frac{4}{f}, \delta+\epsilon=2-\frac{8}{f}$. By $\beta\ne\gamma$, we get $f>12$ and $\beta<\gamma$. By Lemma 10, we have $\delta>\epsilon$. Then $\epsilon<1$. By Lemma 9, we know $\frac{24}{f}=2\alpha+\beta>1$. This implies $f<24$. We give the AVC in Table 10. All possible tilings based on the AVC of Table 10 are deduced as follows. Note that the AVC $\subset\{\alpha\delta\epsilon, \beta\delta\epsilon, \alpha\gamma^2, \beta\gamma^2\}$ in the first row admits no solution satisfying the balance lemma.

Table 9: AVC for Cases $\{\alpha\delta\epsilon, \beta\delta\epsilon, \gamma^3\}$, $\{\alpha\delta\epsilon, \beta\delta\epsilon, \gamma^4\}$ and $\{\alpha\delta\epsilon, \beta\delta\epsilon, \alpha\gamma^3(\beta\gamma^3)\}$.

AVC for $\{\alpha\delta\epsilon, \beta\delta\epsilon, \gamma^3\}$		AVC for $\{\alpha\delta\epsilon, \beta\delta\epsilon, \gamma^4\}$		AVC for $\{\alpha\delta\epsilon, \beta\delta\epsilon, \alpha\gamma^3(\beta\gamma^3)\}$	
f	vertex	f	vertex	f	vertex
all	$\alpha\delta\epsilon, \beta\delta\epsilon, \gamma^3$	all	$\alpha\delta\epsilon, \beta\delta\epsilon, \gamma^4$	all	$\alpha\delta\epsilon, \beta\delta\epsilon, \alpha\gamma^3, \beta\gamma^3$
24	$\alpha^4, \alpha^3\beta, \alpha^2\beta^2, \alpha\beta^3, \beta^4$	16	$\alpha^2\gamma, \alpha\beta\gamma, \beta^2\gamma$	20	$\alpha^2\gamma, \alpha\beta\gamma, \alpha\gamma^3, \beta^2\gamma, \beta\gamma^3, \gamma^2\delta\epsilon, \gamma^5$
36	$\alpha^3\gamma, \alpha^2\beta\gamma, \alpha\beta^2\gamma, \beta^3\gamma$	24	$\alpha^3, \alpha^2\beta, \alpha\beta^2, \beta^3$	36	$\alpha^3, \alpha^2\beta, \alpha\beta^2, \alpha\gamma^3, \beta^3, \beta\gamma^3$
60	$\alpha^2\beta^3, \alpha^3\beta^2$				

Table 10: AVC for Case $\{\alpha\delta\epsilon, \beta\delta\epsilon, \alpha\gamma^2(\beta\gamma^2)\}$.

f	vertex
all	$\alpha\delta\epsilon, \beta\delta\epsilon, \alpha\gamma^2, \beta\gamma^2$
16	$\alpha^4, \alpha^3\beta, \alpha^2\beta^2, \alpha\beta^3, \beta^4$
20	$\alpha^3\gamma, \alpha^2\beta\gamma, \alpha\beta^2\gamma, \beta^3\gamma, \alpha^5, \alpha^4\beta, \alpha^3\beta^2, \alpha^2\beta^3, \alpha\beta^4, \beta^5$

Table 10, $f = 16$.

By the AVC, we know $\beta\gamma\cdots$, $\gamma\delta\cdots$, $\gamma\epsilon\cdots$ and $\epsilon^2\cdots$ are not vertices. If $\alpha^2\beta^2$ is a vertex, we have the AAD for $\alpha^2\beta^2 = |^\beta\alpha^\gamma|^\gamma\alpha^\beta|^\alpha\beta^\delta|^\alpha\beta^\delta|$ or $|^\gamma\alpha^\beta|^\delta\beta^\alpha|^\gamma\alpha^\beta|^\delta\beta^\alpha|$. The first case gives a vertex $\gamma^2\cdots$. By $\alpha\gamma^2$, we get $\gamma^2\cdots = |\alpha|^\epsilon\gamma^\alpha|^\alpha\gamma^\epsilon|$. This implies $\gamma\epsilon\cdots$ is a vertex, contradicting the AVC. The second case gives a vertex $\alpha\gamma\cdots$. By $\alpha\gamma^2$, we get $\alpha\gamma\cdots = |^\gamma\alpha^\beta|^\alpha\gamma^\epsilon|\gamma|$. This implies $\gamma\epsilon\cdots$ is a vertex, contradicting the AVC. Similarly, we know $\alpha\beta^3$, $\alpha^3\beta$, and α^4 are not vertices.

Therefore AVC $\subset \{\alpha\delta\epsilon, \beta\delta\epsilon, \alpha\gamma^2, \beta\gamma^2, \beta^4\}$. We have

$$(\alpha, \beta, \gamma, \delta, \epsilon) = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{4}, \tfrac{3}{2} - \epsilon, \epsilon).$$

By the equations (2.3), (2.4) and (2.6), we get:

$$\cos(\epsilon\pi) = \frac{1}{\sqrt[4]{2}}, \quad \cos a = \sqrt{\sqrt{2} - 1}, \quad \cos b = \sqrt{2}\sqrt{\sqrt{2} - 1} \tag{3.1}$$

In Figure 11, the unique AAD

$$\beta^4 = |^\alpha\beta^\delta|^\alpha\beta^\delta|^\alpha\beta^\delta|^\alpha\beta^\delta|$$

determines T_1, T_2, T_3, T_4. Then $\alpha_2\delta_1\cdots = \alpha_2\delta_1\epsilon_5$ determines T_5. Similarly, we can determine T_6, T_7, T_8. So $\delta_6\epsilon_2\cdots = \alpha_{10}\delta_6\epsilon_2$ or $\beta_{10}\delta_6\epsilon_2$, which induces two tilings in Figure 11 respectively. It is clear that flipping the lower half of one tiling gives the other tiling. Their 3D pictures are given in Figure 7. The first tiling of Figure 11 is the non-symmetric 3-layer earth map tiling $T\{16\alpha\delta\epsilon, 8\beta\gamma^2, 2\beta^4\}$. But there is no $\beta\delta\epsilon$ in this tiling, so it actually belongs to Section 6.

Table 10, $f = 20$.

By the AVC, we know $\gamma^2\cdots$, $\gamma\delta\cdots$, $\gamma\epsilon\cdots$ and $\delta^2\cdots$ are not vertices. If $\beta^3\gamma$ is a vertex, we have the unique AAD for $\beta^3\gamma = |^\alpha\beta^\delta|^\alpha\beta^\delta|^\alpha\beta^\delta|^\alpha\gamma^\epsilon|$. This gives

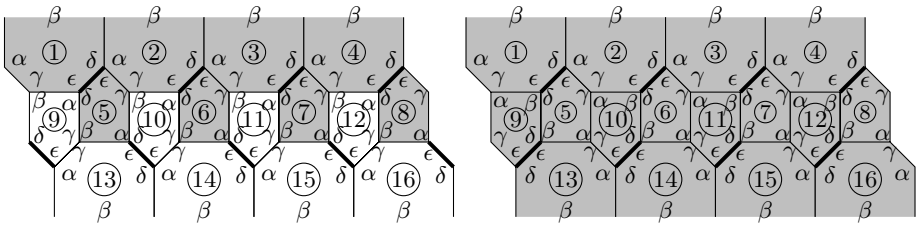


Figure 11: For $f = 16$, $T\{16\alpha\delta\epsilon, 8\beta\gamma^2, 2\beta^4\}$ and $T\{8\alpha\delta\epsilon, 8\beta\delta\epsilon, 8\alpha\gamma^2, 2\beta^4\}$.

a vertex $\alpha\delta\cdots$. By $\alpha\delta\epsilon$, we get $\alpha\delta\cdots = |^\gamma\alpha^\beta|^\beta\delta^\epsilon|^\delta\epsilon^\gamma|$. This implies $\gamma^2\cdots$ is a vertex, a contradiction.

If $\alpha^2\beta\gamma$ is a vertex, we have the unique AAD for $\alpha^2\beta\gamma = |^\gamma\alpha^\beta|^\delta\beta^\alpha|^\delta\beta^\alpha|^\epsilon\gamma^\alpha|$, $|^\beta\alpha^\gamma|^\alpha\beta^\delta|^\alpha\beta^\delta|^\alpha\gamma^\epsilon|$, $|^\gamma\alpha^\beta|^\delta\beta^\alpha|^\epsilon\gamma^\alpha|^\delta\beta^\alpha|$ or $|^\beta\alpha^\gamma|^\alpha\beta^\delta|^\alpha\gamma^\epsilon|^\alpha\beta^\delta|$. The first and second cases give a vertex $\alpha\delta\cdots$. By $\alpha\delta\epsilon$, we get $\alpha\delta\cdots = |^\gamma\alpha^\beta|^\beta\delta^\epsilon|^\delta\epsilon^\gamma|$. This implies $\gamma^2\cdots$ is a vertex, a contradiction. The third and fourth cases give a vertex $\beta\delta\cdots$. By $\beta\delta\epsilon$, we get $\beta\delta\cdots = |^\delta\beta^\alpha|^\beta\delta^\epsilon|^\delta\epsilon^\gamma|$. This implies $\gamma\delta\cdots$ is a vertex, a contradiction.

If $\alpha\beta^2\gamma$ is a vertex, we have the unique AAD for $\alpha\beta^2\gamma = |^\beta\alpha^\gamma|^\beta\alpha^\gamma|^\alpha\beta^\delta|^\alpha\gamma^\epsilon|$, $|^\gamma\alpha^\beta|^\gamma\alpha^\beta|^\delta\beta^\alpha|^\epsilon\gamma^\alpha|$, $|^\gamma\alpha^\beta|^\delta\beta^\alpha|^\gamma\alpha^\beta|^\epsilon\gamma^\alpha|$ or $|^\beta\alpha^\gamma|^\alpha\beta^\delta|^\beta\alpha^\gamma|^\alpha\gamma^\epsilon|$. The first case gives a vertex $\beta\gamma\cdots$. By $\alpha\beta^2\gamma$, we get $\beta\gamma\cdots = |^\beta\alpha^\gamma|^\gamma\alpha^\beta|^\delta\beta^\alpha|^\alpha\gamma^\epsilon|$. This implies $\gamma^2\cdots$ is a vertex, a contradiction. The second, third and fourth cases give a vertex $\beta\delta\cdots$. By $\beta\delta\epsilon$, we get $\beta\delta\cdots = |^\delta\beta^\alpha|^\beta\delta^\epsilon|^\delta\epsilon^\gamma|$. This implies $\gamma\delta\cdots$ is a vertex, a contradiction.

By the previous discussion, we know $\beta\gamma\cdots$ is not a vertex.

If $\alpha^3\gamma$ is a vertex, we have the AAD for $\alpha^3\gamma = |^\alpha\gamma^\epsilon|^\beta\alpha^\gamma|\alpha|$. This implies $\beta\gamma\cdots$ or $\gamma^2\cdots$ is a vertex. This is a contradiction. Similarly, we know $\alpha^5, \alpha^4\beta, \alpha^3\beta^2, \alpha^2\beta^3, \alpha\beta^4, \beta^5$ are not vertices.

Case. $\{\alpha\delta\epsilon, \beta\delta\epsilon, \alpha^2\gamma(\alpha\beta\gamma, \beta^2\gamma)\}$

The angle sums of $\alpha\delta\epsilon, \beta\delta\epsilon, \alpha^2\gamma(\alpha\beta\gamma, \beta^2\gamma)$ and the angle sum for the pentagon imply

$$\alpha = \beta = 1 - \tfrac{4}{f},\ \gamma = \tfrac{8}{f},\ \delta + \epsilon = 1 + \tfrac{4}{f}.$$

We have $1 > \beta > \gamma$. By Lemma 10, this implies $\delta < \epsilon$. By the angle values and $n_4 = n_5$, we have

$$\text{AVC} \subset \{\alpha\delta\epsilon, \beta\delta\epsilon, \alpha^2\gamma, \alpha\beta\gamma, \beta^2\gamma, \alpha\gamma^{\frac{f+4}{8}}, \beta\gamma^{\frac{f+4}{8}}, \gamma^{\frac{f-4}{8}}\delta\epsilon, \gamma^{\frac{f}{4}}\}. \tag{3.2}$$

By the balance lemma, $\alpha\gamma^{\frac{f+4}{8}}, \beta\gamma^{\frac{f+4}{8}}, \gamma^{\frac{f-4}{8}}\delta\epsilon$ or $\gamma^{\frac{f}{4}}$ must appear.

Subcase. $\gamma^{\frac{f}{4}}$ is a vertex.

In Figure 12, the unique AAD of $\gamma^{\frac{f}{4}} = |^\alpha\gamma_1^\epsilon|^\alpha\gamma_{1'}^\epsilon|\cdots$ determines $\frac{f}{4}$ tiles $T_1, T_{1'}, \cdots$. Then $\alpha_1\epsilon\cdots = \alpha_1\delta_2\epsilon$ determines T_2; $\alpha_{1'}\epsilon\cdots = \alpha_{1'}\delta_{2'}\epsilon$ determines $T_{2'}$. Then

$\beta_1\beta_2\cdots=\beta_1\beta_2\gamma_3$ and $\delta_1\epsilon_{2'}\cdots=\alpha_3\delta_1\epsilon_{2'}$ determine T_3; $\alpha_2\epsilon_3\cdots=\alpha_2\delta_4\epsilon_3$ determines T_4. The tiles T_1,T_2,T_3,T_4 together form a **time zone**. Similarly, we can determine $T_{1'},T_{2'},T_{3'},T_{4'}$. By repeating the process, we get the non-symmetric 3-layer earth map tiling $T(f\alpha\delta\epsilon,\frac{f}{2}\beta^2\gamma,2\gamma^{\frac{f}{4}})$. But there is no $\beta\delta\epsilon$ in this tiling, so it actually belongs to Section 6.

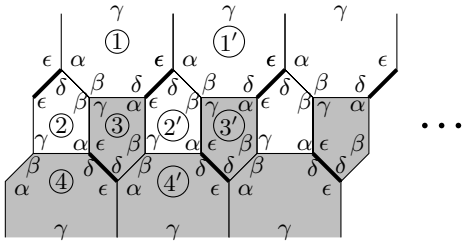


Figure 12: The non-symmetric 3-layer earth map tiling $T(f\,\alpha\delta\epsilon,\frac{f}{2}\,\beta^2\gamma,2\gamma^{\frac{f}{4}})$.

Subcase. $\beta\gamma^{\frac{f+4}{8}}$ **is a vertex.**

In the first row of Figure 13, the unique AAD of $\beta\gamma^{\frac{f+4}{8}}=|^\alpha\gamma_1^\epsilon|^\alpha\gamma_2^\epsilon|\cdots|^\alpha\gamma_3^\epsilon|^\alpha\beta_4^\delta|$ determines $\frac{f+12}{8}$ tiles $T_1,T_2,\cdots,T_3,T_4$. Then $R(\beta_4\gamma_1\cdots)=\gamma^{\frac{f-4}{8}}$ and such $\gamma^{\frac{f-4}{8}}$ determines $\frac{f-4}{8}$ time zones ($\frac{f}{2}-2$ tiles). Then $\alpha_4\epsilon_3\cdots=\alpha_4\delta_5\epsilon_3$ determines T_5. These timezones together with T_1 and T_5 have $\frac{f}{2}$ tiles, and form half of the sphere with boundary vertices $A_1,\cdots,A_{12}$. We have $\beta_5\gamma_4\cdots=\beta^2\gamma$ or $\beta\gamma^{\frac{f+4}{8}}$.

In the left of the second row of Figure 13, $\beta_5\gamma_4\cdots=\beta_5\beta_6\gamma_4$. Then $\alpha_5\cdots=\alpha_5\delta_6\epsilon_7$ determines T_6,T_7. Then $R(\beta\gamma_5\cdots)=\gamma^{\frac{f-4}{8}}$ and such $\gamma^{\frac{f-4}{8}}$ determines $\frac{f-4}{8}$ time zones ($\frac{f}{2}-2$ tiles). Then $\alpha\delta\cdots=\alpha\delta\epsilon_8$ determines T_8. These timezones together with T_4 and T_8 also have $\frac{f}{2}$ tiles, and form exactly the other half of the sphere. Its boundary vertices A_1, $\cdots$, A_{12} can be glued with the A_1, $\cdots$, A_{12} in the first row and form a tiling.

In the right of the second row of Figure 13, $\beta_5\gamma_4\cdots=\beta\gamma^{\frac{f+4}{8}}$. Then $R(\beta_5\gamma_4\cdots)=\gamma^{\frac{f-4}{8}}$ and such $\gamma^{\frac{f-4}{8}}$ determines $\frac{f-4}{8}$ time zones ($\frac{f}{2}-2$ tiles). Then $\alpha_5\epsilon\cdots=\alpha_5\delta_6\epsilon$ determines T_6. These timezones together with T_4 and T_6 also have $\frac{f}{2}$ tiles, and form exactly the other half of the sphere. Its boundary vertices A_1, $\cdots$, A_{12} can be glued with the A_1, $\cdots$, A_{12} in the first row and form another tiling.

These two tilings turn out to be equivalent after a rotation, and give the standard flip modification $T(f\,\alpha\delta\epsilon,\frac{f-4}{2}\beta^2\gamma,4\beta\gamma^{\frac{f+4}{8}})$ (see Figure 14 and 6) of the non-symmetric 3-layer earth map tiling. But there is no $\beta\delta\epsilon$ in this tiling, so it actually belongs to Section 6.

Henceforth, we can assume

$$\text{AVC}\subset\{\alpha\delta\epsilon,\beta\delta\epsilon,\alpha^2\gamma,\alpha\beta\gamma,\beta^2\gamma,\alpha\gamma^{\frac{f+4}{8}},\gamma^{\frac{f-4}{8}}\delta\epsilon\}.$$

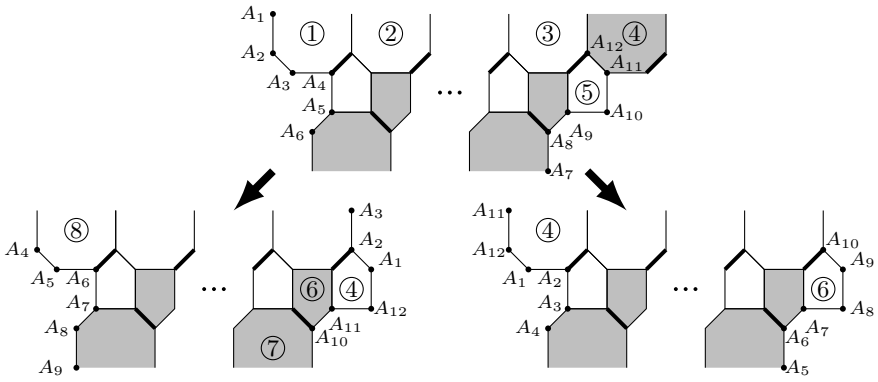


Figure 13: The standard flip modification of the 3-layer earth map tiling.

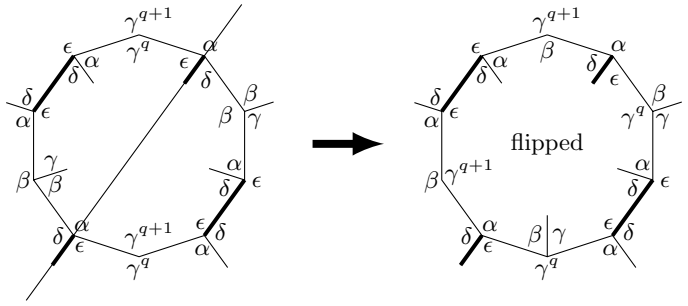


Figure 14: The loop of edges dividing the sphere into two identical halves.

Subcase. $\alpha\gamma^{\frac{f+4}{8}}$ **is a vertex.**

By the AVC, we get that $\epsilon^2\cdots$ is not a vertex. Then we have

$$\begin{aligned}\alpha\gamma^{\frac{f+4}{8}} &= |^\beta\alpha^\gamma|^\epsilon\gamma^\alpha|\cdots|^\epsilon\gamma^\alpha|,\\ &\text{or } |^\beta\alpha^\gamma|^\epsilon\gamma^\alpha|\cdots|^\epsilon\gamma^\alpha|^\alpha\gamma^\epsilon|,\\ &\text{or } |^\beta\alpha^\gamma|^\epsilon\gamma^\alpha|\cdots|^\epsilon\gamma^\alpha|^\alpha\gamma^\epsilon|\cdots|^\alpha\gamma^\epsilon|,\\ &\text{or } |^\beta\alpha^\gamma|^\alpha\gamma^\epsilon|\cdots|^\alpha\gamma^\epsilon|.\end{aligned}$$

• In Figure 15, we have $\alpha\gamma^{\frac{f+4}{8}} = |^\beta\alpha_1^\gamma|^\epsilon\gamma_2^\alpha|\cdots|^\epsilon\gamma_4^\alpha|$, which determines T_1, T_2, T_3, $\cdots$, T_4 ($\frac{f+4}{8}$ tiles). Then $\alpha_2\epsilon_3\cdots = \alpha_2\delta_5\epsilon_3$ determines T_5; $\beta_2\beta_5\cdots = \beta_2\beta_5\gamma_6$ determines T_6; $\alpha_5\epsilon_6\cdots = \alpha_5\delta_7\epsilon_6$ determines T_7; Similarly, $R(\alpha_1\gamma_2\gamma_4\cdots) = \gamma^{\frac{f-12}{8}}$ and such $\gamma^{\frac{f-12}{8}}$ determines $\frac{f-12}{8}$ time zones ($\frac{3f-36}{8}$ tiles). Then $\alpha_6\delta_2\cdots = \alpha_6\delta_2\epsilon_8$ determines T_8.

By $\delta_8\epsilon_2\cdots = |^\alpha\gamma^\epsilon|\cdots|^\alpha\gamma^\epsilon|^\beta\delta^\epsilon|^\delta\epsilon^\gamma|$, we can determine $T_9, \cdots, T_{10}$, altogether determining $\frac{f-4}{8}$ tiles. Then $\alpha_2\epsilon_9\cdots = \alpha_2\delta_{11}\epsilon_9$ determines T_{11}; $\beta_9\beta_{11}\cdots = \beta_9\beta_{11}\gamma_{12}$ determines T_{12}; $\alpha_{11}\epsilon_{12}\cdots = \alpha_{11}\delta_{13}\epsilon_{12}$ determines T_{13}. Then $R(|^\beta\delta_8^\epsilon|^\delta\epsilon_2^\gamma|^\alpha\gamma_1^\epsilon|\cdots) = \gamma^{\frac{f-4}{8}}$ and such $\gamma^{\frac{f-4}{8}}$ determines $\frac{f-4}{8}$ time zones ($(\frac{f}{2}-2)$ tiles). Then $\alpha_4\gamma_{17}\cdots = \alpha_4\beta_1\cdots = \alpha_4\beta_1\gamma_{17}$ determines T_{17}; $\beta_4\epsilon_{17}\cdots = \beta_4\delta_{18}\epsilon_{17}$ determines T_{18}.

More repetitions give the unique tiling with the AVC $\{4\alpha\beta\gamma, (f-6)\alpha\delta\epsilon, (\frac{f}{2}-4)\beta^2\gamma, 4\beta\delta\epsilon, 2\alpha\gamma^{\frac{f+4}{8}}, 2\gamma^{\frac{f-4}{8}}\delta\epsilon\}$. In Figure 15, T_4, T_{17}, T_{18}, T_{19} form a UFO (see Page 8 for its definition and Figure 8 and 9).

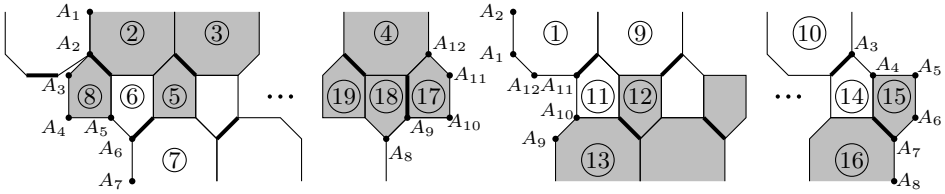


Figure 15: $\alpha\gamma^{\frac{f+4}{8}} = |^\beta\alpha^\gamma|^\epsilon\gamma^\alpha|\cdots|^\epsilon\gamma^\alpha|$.

When $f = 20$, Figure 15 gives the second tiling with AVC 1 in Figure 9.

• In Figure 16, we have $\alpha\gamma^{\frac{f+4}{8}} = |\alpha_1|^\epsilon\gamma_2^\alpha|\cdots|^\epsilon\gamma_3^\alpha|^\alpha\gamma_4^\epsilon|$, which determines $T_1, \cdots, T_4$ ($\frac{f+12}{8}$ tiles). Just like in Figure 15, we can also determine all tiles except T_6, T_7, T_8, T_9, which form a UFO and can be flipped.

One tiling is $T(2\alpha^2\gamma, 4\alpha\beta\gamma, (f-10)\alpha\delta\epsilon, (\frac{f}{2}-6)\beta^2\gamma, 8\beta\delta\epsilon, 2\alpha\gamma^{\frac{f+4}{8}}, 2\gamma^{\frac{f-4}{8}}\delta\epsilon)$. Figure 16 shows a tiling with three UFOs: $\{T_4, T_8, T_9, T_{10}\}$, $\{T_6, T_7, T_8, T_9\}$ and $\{T_1, T_4, T_{10}, T_{11}\}$.

The other tiling has the AVC $\{2\alpha^2\gamma, 6\alpha\beta\gamma, (f-11)\alpha\delta\epsilon, (\frac{f}{2}-7)\beta^2\gamma, 8\beta\delta\epsilon, \alpha\gamma^{\frac{f+4}{8}}, 3\gamma^{\frac{f-4}{8}}\delta\epsilon\}$. Flipping $\{T_6, T_7, T_8, T_9\}$ in Figure 16 gives three UFOs: $\{T_3, T_5, T_8, T_9\}$, $\{T_6, T_7, T_8, T_9\}$ and $\{T_1, T_4, T_{10}, T_{11}\}$.

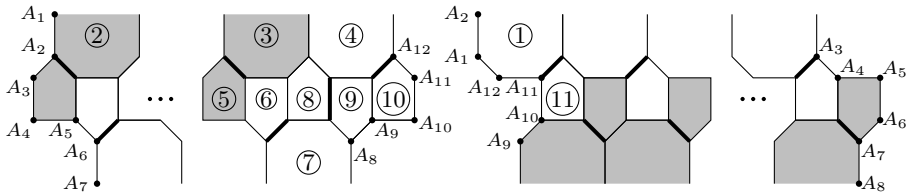


Figure 16: $\alpha\gamma^{\frac{f+4}{8}} = |^{\beta}\alpha^{\gamma}|^{\epsilon}\gamma^{\alpha}|\cdots|^{\epsilon}\gamma^{\alpha}|^{\alpha}\gamma^{\epsilon}|$.

When $f = 20$, Figure 16 gives the tiling with AVC 3 and the first tiling with AVC 5 in Figure 9.

• In Figure 17, we have $\alpha\gamma^{\frac{f+4}{8}} = |\alpha_1|^{\epsilon}\gamma_2^{\alpha}|\cdots|^{\epsilon}\gamma_3^{\alpha}|^{\alpha}\gamma_4^{\epsilon}|\cdots|^{\alpha}\gamma_5^{\epsilon}|$, which determines $T_1, T_2, \cdots, T_3, T_4, \cdots, T_5$ ($\frac{f+12}{8}$ tiles). Note that the position of γ_4 can be any place not next to α_1. Then just like in Figure 15, we can also determine all tiles except T_7, T_8, T_9, T_{10}, which form a UFO and can be flipped. So we get many tilings with the same AVC $\{2\alpha^2\gamma, 6\alpha\beta\gamma, (f-11)\alpha\delta\epsilon, (\frac{f}{2}-7)\beta^2\gamma, 8\beta\delta\epsilon, \alpha\gamma^{\frac{f+4}{8}}, 3\gamma^{\frac{f-4}{8}}\delta\epsilon\}$ and $T(2\alpha^2\gamma, 5\alpha\beta\gamma, (f-10)\alpha\delta\epsilon, (\frac{f}{2}-6)\beta^2\gamma, 7\beta\delta\epsilon, \alpha\gamma^{\frac{f+4}{8}}, 3\gamma^{\frac{f-4}{8}}\delta\epsilon)$. Figure 17 gives a tiling with three UFO: $\{T_7, T_8, T_9, T_{10}\}$, $\{T_4, T_9, T_{10}, T_{11}\}$ and $\{T_1, T_5, T_{14}, T_{15}\}$. After flipping the UFO $\{T_7, T_8, T_9, T_{10}\}$, the new tiling still has three UFO and the other two UFO are: $\{T_3, T_6, T_9, T_{10}\}$ and $\{T_1, T_5, T_{14}, T_{15}\}$.

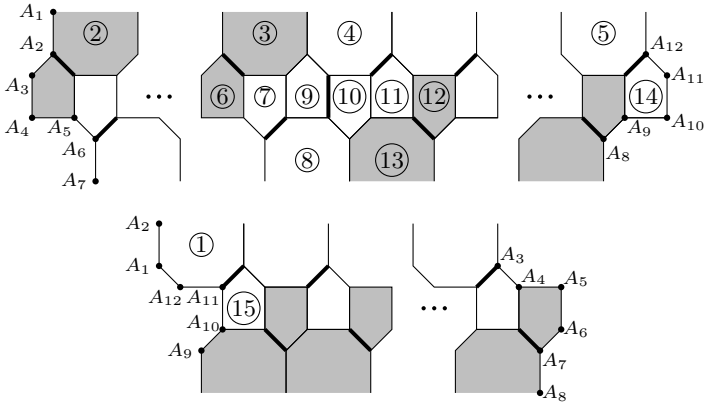


Figure 17: $\alpha\gamma^{\frac{f+4}{8}} = |^{\beta}\alpha^{\gamma}|^{\epsilon}\gamma^{\alpha}|\cdots|^{\epsilon}\gamma^{\alpha}|^{\alpha}\gamma^{\epsilon}|\cdots|^{\alpha}\gamma^{\epsilon}|$.

When $f = 20$, Figure 17 gives the tiling with AVC 4 and the second tiling with AVC 5 in Figure 9.

• In Figure 18, we have $\alpha\gamma^{\frac{f+4}{8}} = |\alpha_1|^{\alpha}\gamma_2^{\epsilon}|^{\alpha}\gamma_3^{\epsilon}|\cdots|^{\alpha}\gamma_4^{\epsilon}|$, which determines $T_2, T_3, \cdots, T_4$ ($\frac{f+4}{8}$ tiles). Just like in Figure 15, we can also determine $T_5, \cdots, T_9$. We have $\alpha_2\gamma_1\cdots = \alpha^2\gamma, \alpha\beta\gamma$ or $\alpha\gamma^{\frac{f+4}{8}}$.

When $\alpha_2\gamma_1\cdots=\alpha^2\gamma$, if $\alpha\alpha_2\gamma_1=|^\beta\alpha^\gamma|^\epsilon\gamma_1^\alpha|^\gamma\alpha_2^{\beta_2}|$, this gives a vertex $\beta_2^\alpha|^\alpha\beta=|^\delta\beta^\alpha|^\alpha\beta^\delta|^\alpha\gamma^\epsilon|$. This gives a vertex $\delta|\epsilon\cdots$, contradicting the AVC. Therefore $\alpha\alpha_2\gamma_1=|^\gamma\alpha^\beta|^\epsilon\gamma_1^\alpha|^\gamma\alpha_2^{\beta_2}|$. This gives a vertex $\beta_2^\alpha|^\alpha\gamma\cdots=|\alpha|^\delta\beta^\alpha|^\alpha\gamma^\epsilon|$. Then we have the AAD $\delta_2\epsilon_5\cdots=|^\delta\beta^\alpha|^\beta\delta^\epsilon\rceil^\delta\epsilon^\gamma|$ or $\gamma^{\frac{f-4}{8}}\delta\epsilon=|^\epsilon\gamma^\alpha|\cdots|^\epsilon\gamma^\alpha|^\beta\delta^\epsilon\rceil^\delta\epsilon^\gamma|$. Hence we get $\beta_7\gamma_5\cdots=\beta\gamma\delta\cdots$ or $\beta\gamma\epsilon\cdots$, contradicting the AVC.

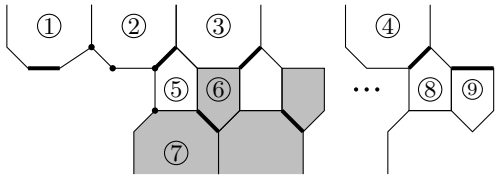


Figure 18: $\alpha\gamma^{\frac{f+4}{8}}=|^\beta\alpha^\gamma|^\alpha\gamma^\epsilon|\cdots|^\alpha\gamma^\epsilon|$ and $\alpha_2\gamma_1\cdots=\alpha^2\gamma$.

When $\alpha_2\gamma_1\cdots=\alpha\beta\gamma$, just like in Figure 15, we can also determine all tiles in Figure 19 which gives a tiling with the AVC $\{4\alpha\beta\gamma,(f-6)\alpha\delta\epsilon,(\frac{f}{2}-4)\beta^2\gamma,4\beta\delta\epsilon,2\alpha\gamma^{\frac{f+4}{8}},2\gamma^{\frac{f-4}{8}}\delta\epsilon\}$. It has one UFO: $\{T_1,\ T_4,\ T_8,\ T_9\}$.

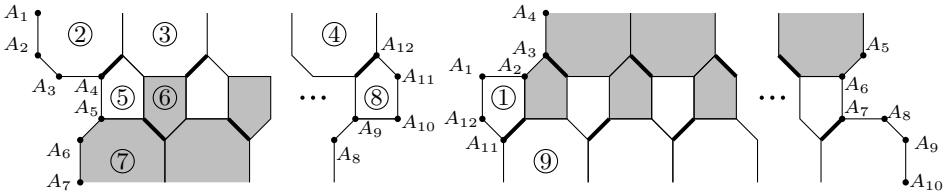


Figure 19: $\alpha\gamma^{\frac{f+4}{8}}=|^\beta\alpha^\gamma|^\alpha\gamma^\epsilon|\cdots|^\alpha\gamma^\epsilon|$ and $\alpha_2\gamma_1\cdots=\alpha\beta\gamma$.

When $f=20$, Figure 19 gives the first tiling with AVC 1 in Figure 9.

When $\alpha_2\gamma_1\cdots=\alpha\gamma^{\frac{f+4}{8}}$, just like in Figure 15, we can also determine all tiles in Figure 20 which gives a tiling with the AVC $\{4\alpha\beta\gamma,(f-6)\alpha\delta\epsilon,(\frac{f}{2}-4)\beta^2\gamma,4\beta\delta\epsilon,2\alpha\gamma^{\frac{f+4}{8}},2\gamma^{\frac{f-4}{8}}\delta\epsilon\}$. It has two UFOs: $\{T_1,\ T_4,\ T_8,\ T_9\}$ and $\{T_2,\ T_5,\ T_{10},\ T_{11}\}$.

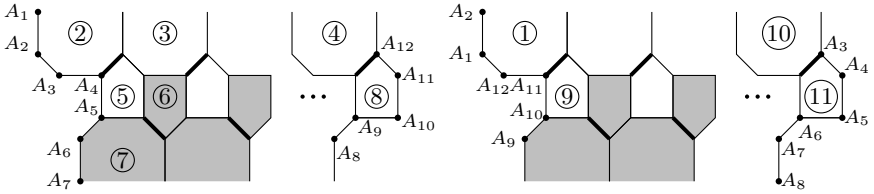


Figure 20: $\alpha\gamma^{\frac{f+4}{8}}=|^\beta\alpha^\gamma|^\alpha\gamma^\epsilon|\cdots|^\alpha\gamma^\epsilon|$ and $\alpha_2\gamma_1\cdots=\alpha\gamma^{\frac{f+4}{8}}$.

When $f=20$, Figure 20 gives the third tiling with AVC 1 in Figure 9.

Henceforth, we can assume

$$\text{AVC} \subset \{\alpha\delta\epsilon, \beta\delta\epsilon, \alpha^2\gamma, \alpha\beta\gamma, \beta^2\gamma, \gamma^{\frac{f-4}{8}}\delta\epsilon\}.$$

Subcase. $\gamma^{\frac{f-4}{8}}\delta\epsilon$ is a vertex.

By the AVC, we get that $\epsilon^2\cdots$ is not a vertex. Then we have four possible AAD for $\gamma^{\frac{f-4}{8}}\delta\epsilon$:

$$\begin{aligned}
\gamma^{\frac{f-4}{8}}\delta\epsilon &= |^{\alpha}\gamma^{\epsilon}|\cdots|^{\alpha}\gamma^{\epsilon}|^{\beta}\delta^{\epsilon}\rceil^{\delta}\epsilon^{\gamma}|,\\
&\text{or } |^{\epsilon}\gamma^{\alpha}|\cdots|^{\epsilon}\gamma^{\alpha}|^{\beta}\delta^{\epsilon}\rceil^{\delta}\epsilon^{\gamma}|,\\
&\text{or } |^{\epsilon}\gamma^{\alpha}|\cdots|^{\epsilon}\gamma^{\alpha}|^{\alpha}\gamma^{\epsilon}|^{\beta}\delta^{\epsilon}\rceil^{\delta}\epsilon^{\gamma}|,\\
&\text{or } |^{\epsilon}\gamma^{\alpha}|\cdots|^{\epsilon}\gamma^{\alpha}|^{\alpha}\gamma^{\epsilon}|\cdots|^{\alpha}\gamma^{\epsilon}|^{\beta}\delta^{\epsilon}\rceil^{\delta}\epsilon^{\gamma}|.
\end{aligned}$$

• In Figure 21, we have $\gamma^{\frac{f-4}{8}}\delta\epsilon = |^{\alpha}\gamma_3^{\epsilon}|\cdots|^{\alpha}\gamma_4^{\epsilon}|^{\beta}\delta_1^{\epsilon}\rceil^{\delta}\epsilon_2^{\gamma}|$, which determines $T_1, T_2, T_3, \cdots, T_4$ ($\frac{f+12}{8}$ tiles). Just like in Figure 15, we can also determine all tiles in Figure 21 which gives the tiling $T(2\alpha^2\gamma, 6\alpha\beta\gamma, (f-10)\alpha\delta\epsilon, (\frac{f}{2}-6)\beta^2\gamma, 6\beta\delta\epsilon, 4\gamma^{\frac{f-4}{8}}\delta\epsilon)$. It has one UFO: $\{T_5, T_6, T_7, T_8\}$.

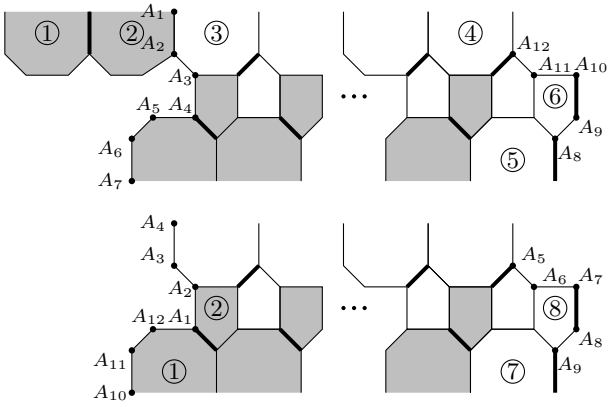


Figure 21: $\gamma^{\frac{f-4}{8}}\delta\epsilon = |^{\alpha}\gamma^{\epsilon}|\cdots|^{\alpha}\gamma^{\epsilon}|^{\beta}\delta^{\epsilon}\rceil^{\delta}\epsilon^{\gamma}|$.

When $f = 20$, Figure 21 gives the tiling with AVC 2 in Figure 9.

• For the other AAD of $\gamma^{\frac{f-4}{8}}\delta\epsilon$, half of the sphere can be determined through AAD. As shown in Figures 22, 23, 24, we get three different half spheres HS1, HS2, HS3 from $|\delta_1\rceil\epsilon_2|^{\epsilon}\gamma_3^{\alpha}|\cdots = |\delta_1\rceil\epsilon_2|^{\epsilon}\gamma_3^{\alpha}|\cdots|^{\epsilon}\gamma_4^{\alpha}|$, or $|\delta_1\rceil\epsilon_2|^{\epsilon}\gamma_3^{\alpha}|\cdots|^{\epsilon}\gamma_4^{\alpha}|^{\alpha}\gamma_5^{\epsilon}|$, or $|\delta_1\rceil\epsilon_2|^{\epsilon}\gamma_3^{\alpha}|\cdots|^{\epsilon}\gamma_4^{\alpha}|^{\alpha}\gamma_5^{\epsilon}|\cdots|^{\alpha}\gamma_6^{\epsilon}|$ respectively, but with the same boundary and the same labeling of $A_1, \cdots, A_{12}$. Similarly, the remaining half of the sphere can also be determined by the AAD.

When $f \geq 28$, each of HS1, HS2, HS3 with boundary vertices $A_1, \cdots, A_{12}$ outside the parenthesis can be glued with any of HS1, HS2, HS3, including itself, by matching the $A_1, \cdots, A_{12}$ inside the parenthesis.

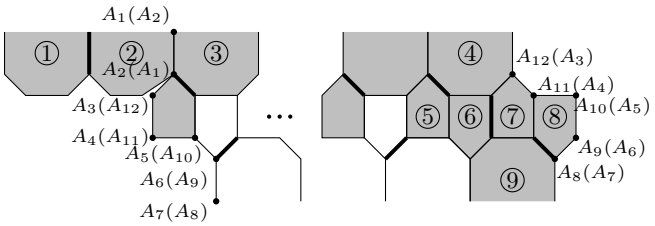


Figure 22: The half sphere HS1 from $\gamma^{\frac{f-4}{8}}\delta\epsilon = |^{\epsilon}\gamma^{\alpha}|\cdots|^{\epsilon}\gamma^{\alpha}|^{\beta}\delta^{\epsilon}\rfloor^{\delta}\epsilon^{\gamma}|$.

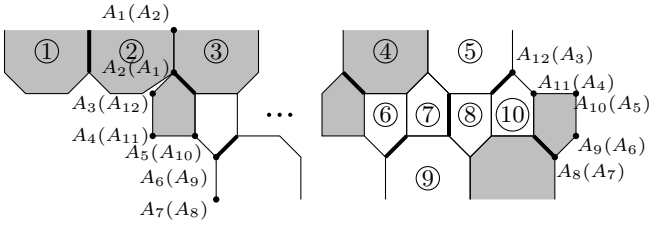


Figure 23: The half sphere HS2 from $\gamma^{\frac{f-4}{8}}\delta\epsilon = |^{\epsilon}\gamma^{\alpha}|\cdots|^{\epsilon}\gamma^{\alpha}|^{\alpha}\gamma^{\epsilon}|^{\beta}\delta^{\epsilon}\rfloor^{\delta}\epsilon^{\gamma}|$.

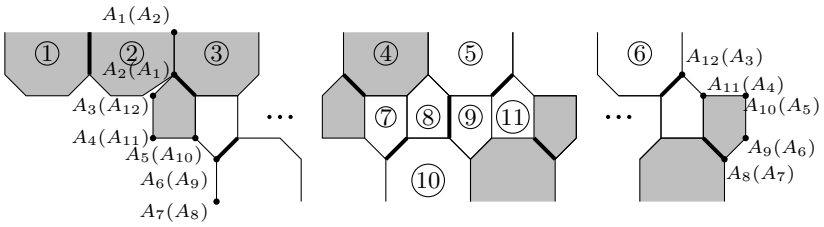


Figure 24: The half sphere HS3 from $\gamma^{\frac{f-4}{8}}\delta\epsilon = |^{\epsilon}\gamma^{\alpha}|\cdots|^{\alpha}\gamma^{\epsilon}|\cdots|^{\alpha}\gamma^{\epsilon}|^{\beta}\delta^{\epsilon}\rfloor^{\delta}\epsilon^{\gamma}|$.

There are two UFOs: $\{T_4, T_5, T_6, T_7\}$ and $\{T_6, T_7, T_8, T_9\}$ in HS1. There are two UFOs: $\{T_6, T_7, T_8, T_9\}$ and $\{T_5, T_7, T_8, T_{10}\}$ in HS2. There are two UFOs: $\{T_7, T_8, T_9, T_{10}\}$ and $\{T_5, T_8, T_9, T_{11}\}$ in HS3. Whatever tiling is formed by gluing these half spheres, there are always four UFOs in the end.

When $f = 20$, there is no HS3. Let HS2$'$ denote the modification of HS2 by flipping its UFO $\{T_6, T_7, T_8, T_9\}$. Then HS1+HS1=HS2$'$+HS2$'$ gives the first tiling with AVC 6 in Figure 9; and HS1+HS2=HS2+HS2$'$ gives the tiling with AVC 7 in Figure 9; and HS1+HS2$'$ gives the second tiling with AVC 6 in Figure 9; and HS2+HS2 gives the tiling with AVC 8 in Figure 9.

In conclusion, Case. $\{\alpha\delta\epsilon, \beta\delta\epsilon, \alpha^2\gamma(\alpha\beta\gamma, \beta^2\gamma)\}$ admits various modifications of the non-symmetric 3-layer earth map tiling with $f = 8k + 4$, $k \geq 2$, summarized in Table 3. In particular, we count the number of different tilings associated to each possible AVC explicitly in this table. □

4 Tilings with a unique a^2b-vertex type $\alpha\delta^2$, $\beta\delta^2$ or $\gamma\delta^2$

Our strategy in the following sections is to first tile the partial neighborhood of a special tile from Lemma 4.

Proposition 3. *There is no non-symmetric a^4b-tiling with general angles, such that $\alpha\delta^2$, $\beta\delta^2$ or $\gamma\delta^2$ is the unique a^2b-vertex type.*

Proof. Since ϵ can not appear in an a^2b-vertex (thus not in any degree 3 vertex), the special tile must have a vertex of degree 4 or 5 at ϵ and must have degree 3 vertices at the other four corners (see Figure 25).

By Lemma 8 and the parity lemma, we know that $\delta\epsilon^3$ or ϵ^4 is a vertex.

If $\alpha\delta^2$ is the unique a^2b-vertex type, the partial neighborhood of a special tile is shown in the first and second pictures of Figure 25. In both pictures $\delta_1 \cdots = \alpha_3\delta_1\delta_2$ determines T_2. These two pictures show two possible arrangements of T_3. In the first picture the degree 3 vertex $\beta_1\gamma_3 \cdots = \alpha\beta\gamma$, $\beta^2\gamma$ or $\beta\gamma^2$. In the second picture the degree 3 vertex $\beta_1\beta_3 \cdots = \alpha\beta^2$, β^3 or $\beta^2\gamma$. So three vertices $\alpha\delta^2$ + one of $\{\delta\epsilon^3, \epsilon^4\}$ + one of $\{\alpha\beta^2, \alpha\beta\gamma, \beta^3, \beta^2\gamma, \beta\gamma^2\}$ must appear.

If $\beta\delta^2$ is the unique a^2b-vertex type, the partial neighborhood of a special tile is shown in the third and fourth pictures of Figure 25. In both pictures $\delta_1 \cdots = \beta_3\delta_1\delta_2$ determines T_2. These two pictures show two possible arrangements of T_3. Then we have the degree 3 vertex $\beta_1\delta_3 \cdots = \beta_1\delta_3\delta_4$, which determines T_4. In the third and fourth pictures the degree 3 vertex $\alpha\beta \cdots = \alpha^2\beta$, $\alpha\beta^2$ or $\alpha\beta\gamma$. So three vertices $\beta\delta^2$ + one of $\{\delta\epsilon^3, \epsilon^4\}$ + one of $\{\alpha^2\beta, \alpha\beta\gamma, \alpha\beta^2\}$ must appear.

If $\gamma\delta^2$ is the unique a^2b-vertex type, the partial neighborhood of a special tile is shown in the fifth picture of Figure 25. In this picture $\delta_1 \cdots = \gamma_3\delta_1\delta_2$ determines T_2, T_3. The degree 3 vertex $\alpha_3\beta_1 \cdots = \alpha^2\beta$, $\alpha\beta^2$ or $\alpha\beta\gamma$. So three vertices $\gamma\delta^2$ + one of $\{\delta\epsilon^3, \epsilon^4\}$ + one of $\{\alpha^2\beta, \alpha\beta^2, \alpha\beta\gamma\}$ must appear.

For each combination of 3 vertices, we will apply the irrational angle lemma and AAD to get Table 11. The proof is completed by the contradictions in the

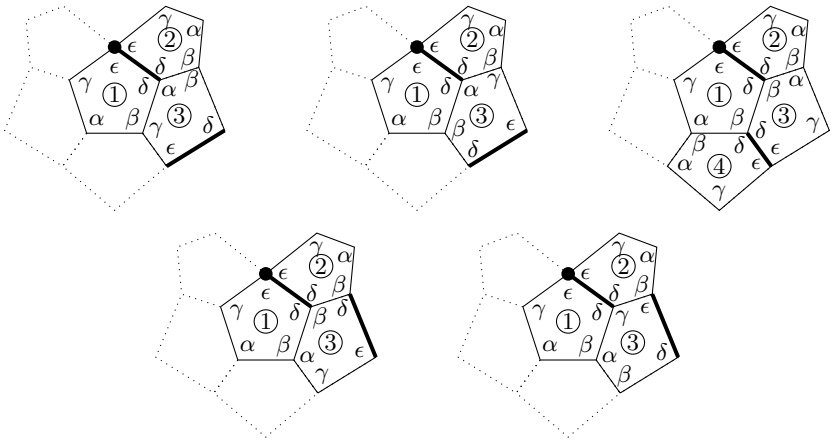


Figure 25: $\alpha\delta^2$, $\beta\delta^2$ or $\gamma\delta^2$ is the unique a^2b-vertex type.

last column of Table 11, similar to Proposition 1.

Table 11: All a^2b-vertices are $\alpha\delta^2, \beta\delta^2$ or $\gamma\delta^2$.

Vertex			Angles	Irrational Angle Lemma	AAD	Contradiction
$\alpha\delta^2$	$\delta\epsilon^3$	$\alpha\beta^2$	$(2-2\delta, \delta, \frac{1}{3}+\frac{1}{3}\delta+\frac{4}{7}, \delta, \frac{2}{3}-\frac{1}{3}\delta)$	$6n_1+n_5=3n_2+n_3+3n_4$	$\lvert\gamma\alpha^\beta\rvert\beta\delta^\epsilon\rvert\epsilon\delta^\beta\rvert$	No $\beta\gamma\cdots$ by the irrational angle lemma
		$\alpha\beta\gamma$	$(1-\frac{12}{7}, 1-\gamma+\frac{12}{7}, \gamma, \frac{1}{2}+\frac{6}{7}, \frac{1}{2}-\frac{2}{7})$	$n_2=n_3$	$\lvert\epsilon\delta^\beta\rvert\gamma\epsilon^\delta\rvert\delta\epsilon\gamma\rvert\gamma\epsilon^\delta\rvert$	No $\gamma^2\cdots$
		β^3	$(2-2\delta, \frac{2}{3}, -\frac{1}{3}+\frac{1}{3}\delta+\frac{4}{7}, \delta, \frac{2}{3}-\frac{1}{3}\delta)$	$6n_1+n_5=4n_3+3n_4$		
		$\beta^2\gamma$	$(2-2\delta, \frac{5}{3}-\frac{4}{3}\delta-\frac{4}{7}, -\frac{4}{3}+\frac{8}{3}\delta+\frac{8}{7}, \delta, \frac{2}{3}-\frac{1}{3}\delta)$	$6n_1+4n_2+n_5=8n_3+3n_4$		
		$\beta\gamma^2$	$(2-2\delta, -\frac{4}{3}+\frac{8}{3}\delta+\frac{8}{7}, \frac{5}{3}-\frac{4}{3}\delta-\frac{4}{7}, \delta, \frac{2}{3}-\frac{1}{3}\delta)$	$6n_1+4n_3+n_5=8n_2+3n_4$	$\lvert\gamma\alpha^\beta\rvert\beta\delta^\epsilon\rvert\epsilon\delta^\beta\rvert$	No $\beta^2\cdots$
	ϵ^4	$\alpha\beta^2$	$(2-2\delta, \delta, \frac{1}{2}+\frac{4}{7}, \delta, \frac{1}{2})$	$2n_1=n_2+n_4$		No $\beta\gamma\cdots$
		β^3	$(2-2\delta, \frac{2}{3}, -\frac{1}{6}+\delta+\frac{4}{7}, \delta, \frac{1}{2})$	$2n_1=n_3+n_4$		
		$\alpha\beta\gamma$	$(1-\frac{8}{7}, 1-\gamma+\frac{8}{7}, \gamma, \frac{1}{2}+\frac{4}{7}, \frac{1}{2})$	$n_2=n_3$	$\lvert\delta\epsilon\gamma\rvert\gamma\epsilon^\delta\rvert\delta\epsilon\gamma\rvert\gamma\epsilon^\delta\rvert$	No $\gamma^2\cdots$
		$\beta^2\gamma$	$(2-2\delta, \frac{3}{2}-\delta-\frac{4}{7}, -1+2\delta+\frac{8}{7}, \delta, \frac{1}{2})$	$2n_1+n_2=2n_3+n_4$		
		$\beta\gamma^2$	$(2-2\delta, -1+2\delta+\frac{8}{7}, \frac{3}{2}-\delta-\frac{4}{7}, \delta, \frac{1}{2})$	$2n_1+n_3=2n_2+n_4$	$\lvert\gamma\alpha^\beta\rvert\beta\delta^\epsilon\rvert\epsilon\delta^\beta\rvert$	No $\beta^2\cdots$
$\beta\delta^2$	$\delta\epsilon^3$	$\alpha^2\beta$	$(\delta, 2-2\delta, \frac{1}{3}+\frac{1}{3}\delta+\frac{4}{7}, \delta, \frac{2}{3}-\frac{1}{3}\delta)$	$3n_1+n_3+3n_4=6n_2+n_5$	$\lvert\epsilon\delta^\beta\rvert\gamma\epsilon^\delta\rvert\delta\epsilon\gamma\rvert\gamma\epsilon^\delta\rvert$	No $\beta\gamma\cdots$
		$\alpha\beta^2$	$(-2+4\delta, 2-2\delta, \frac{t}{3}-\frac{8}{3}\delta+\frac{4}{7}, \delta, \frac{2}{3}-\frac{1}{3}\delta)$	$12n_1+3n_4=6n_2+8n_3+n_5$		No $\gamma^2\cdots$
		$\alpha\beta\gamma$	$(1-\gamma+\frac{12}{7}, 1-\frac{12}{7}, \gamma, \frac{1}{2}+\frac{6}{7}, \frac{1}{2}-\frac{2}{7})$	$n_1=n_3$		
	ϵ^4	$\alpha^2\beta$	$(\delta, 2-2\delta, \frac{1}{2}+\frac{4}{7}, \delta, \frac{1}{2})$	$n_1+n_4=2n_2$	$\lvert\delta\epsilon\gamma\rvert\gamma\epsilon^\delta\rvert\delta\epsilon\gamma\rvert\gamma\epsilon^\delta\rvert$	
		$\alpha\beta^2$	$(-2+4\delta, 2-2\delta, \frac{5}{2}-3\delta+\frac{4}{7}, \delta, \frac{1}{2})$	$4n_1+n_4=2n_2+3n_3$		
		$\alpha\beta\gamma$	$(1-\gamma+\frac{8}{7}, 1-\frac{8}{7}, \gamma, \frac{1}{2}+\frac{4}{7}, \frac{1}{2})$	$n_1=n_3$		
$\gamma\delta^2$	$\delta\epsilon^3$	$\alpha^2\beta$	$(\frac{5}{3}-\frac{4}{3}\delta-\frac{4}{7}, -\frac{4}{7}+\frac{8}{3}\delta+\frac{8}{7}, 2-2\delta, \delta, \frac{2}{3}-\frac{1}{3}\delta)$	$4n_1+6n_3+n_5=8n_2+3n_4$	$\lvert\epsilon\delta^\beta\rvert\gamma\epsilon^\delta\rvert\delta\epsilon\gamma\rvert\gamma\epsilon^\delta\rvert$	
		$\alpha\beta^2$	$(-\frac{4}{3}+\frac{8}{3}\delta+\frac{8}{7}, \frac{5}{3}-\frac{4}{3}\delta-\frac{4}{7}, 2-2\delta, \delta, \frac{2}{3}-\frac{1}{3}\delta)$	$8n_1+3n_4=4n_2+6n_3+n_5$		
		$\alpha\beta\gamma$	$(1-\beta+\frac{12}{7}, \beta, 1-\frac{12}{7}, \frac{1}{2}+\frac{6}{7}, \frac{1}{2}-\frac{2}{7})$	$n_1=n_2$	$\lvert\alpha\gamma^\epsilon\rvert\beta\delta^\epsilon\rvert\epsilon\delta^\beta\rvert$	No $\beta\epsilon\cdots$
	ϵ^4	$\alpha^2\beta$	$(\frac{3}{2}-\delta-\frac{4}{7}, -1+2\delta+\frac{8}{7}, 2-2\delta, \delta, \frac{1}{2})$	$n_1+2n_3=2n_2+n_4$	$\lvert\delta\epsilon\gamma\rvert\gamma\epsilon^\delta\rvert\delta\epsilon\gamma\rvert\gamma\epsilon^\delta\rvert$	No $\gamma^2\cdots$
		$\alpha\beta^2$	$(-1+2\delta+\frac{8}{7}, \frac{3}{2}-\delta-\frac{4}{7}, 2-2\delta, \delta, \frac{1}{2})$	$2n_1+n_4=n_2+2n_3$		
		$\alpha\beta\gamma$	$(1-\beta+\frac{8}{7}, \beta, 1-\frac{8}{7}, \frac{1}{2}+\frac{4}{7}, \frac{1}{2})$	$n_1=n_2$	$\lvert\alpha\gamma^\epsilon\rvert\beta\delta^\epsilon\rvert\epsilon\delta^\beta\rvert$	No $\beta\epsilon\cdots$

□

5 Tilings with a unique a^2b-vertex type $\beta\delta\epsilon$

We divide the discussion into five cases in Figure 26, according to the location of the vertex H of a special tile (indicated by $\bullet$). The degree of H is $3, 4$ or 5.

Proposition 4. *There is no non-symmetric a^4b-tiling with general angles, such that $\beta\delta\epsilon$ is the unique a^2b-vertex type and $\alpha^2\beta$ is a vertex.*

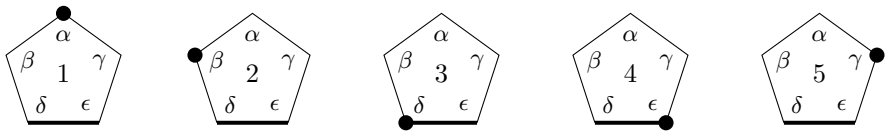


Figure 26: Special tile for the edge combination a^4b.

Proof. By Lemma 8, one of $\{\alpha\beta\gamma, \alpha^2\gamma, \alpha\gamma^2, \beta^2\gamma, \beta\gamma^2, \gamma^3, \alpha\gamma^3, \beta\gamma^3, \gamma^4, \gamma^5\}$ must appear. In any of these cases, we have $n_4 = n_5$ by the irrational angle lemma.

If there is no degree ≥ 4 vertex $\delta\epsilon\cdots$, by $\beta\delta\epsilon$ being the unique a^2b-vertex type and $n_4 = n_5$, we get $\delta\cdots = \delta\epsilon\cdots = \beta\delta\epsilon$. Then $\#\beta \geq \#\beta\delta\epsilon + \#\alpha^2\beta = f + \#\alpha^2\beta > f$, a contradiction. Therefore, we know degree ≥ 4 vertex $\delta\epsilon\cdots$ must appear.

For each case of $\{\alpha\beta\gamma, \alpha\gamma^2, \beta^2\gamma, \beta\gamma^2, \gamma^3, \alpha\gamma^3, \beta\gamma^3, \gamma^4, \gamma^5\}$, we summarize the computations of angle values in Table 12, excluding the symmetric ($\beta = \gamma$) Case $\{\beta\delta\epsilon, \alpha^2\beta, \alpha^2\gamma\}$.

Case. $\{\beta\delta\epsilon, \alpha^2\beta, \gamma^3\}$

The angle sums of $\beta\delta\epsilon, \alpha^2\beta, \gamma^3$ and the angle sum for the pentagon imply

$$\alpha = \tfrac{1}{3} + \tfrac{4}{f},\ \beta = \tfrac{4}{3} - \tfrac{8}{f},\ \gamma = \tfrac{2}{3},\ \delta + \epsilon = \tfrac{2}{3} + \tfrac{8}{f}.$$

By the angle values and $n_4 = n_5$, we get $\delta\cdots = \beta\delta\epsilon$, $\alpha^2\delta\epsilon$, $\alpha\gamma\delta\epsilon$, $\delta^2\epsilon^2$ or $\alpha^3\delta\epsilon$.

If $\alpha^2\delta\epsilon$ is a vertex, then AVC $\subset \{\beta\delta\epsilon, \alpha^2\beta, \gamma^3, \alpha^4, \alpha^2\delta\epsilon, \delta^2\epsilon^2\}$. We have the AAD $|^\epsilon\delta^\beta|\alpha|$ at $\alpha^2\delta\epsilon$. This gives a vertex $\beta^2\cdots$ or $\beta\gamma\cdots$, contradicting the AVC. Similarly, $\delta^2\epsilon^2$ and $\alpha^3\delta\epsilon$ are not vertices.

If $\alpha\gamma\delta\epsilon$ is a vertex, then AVC $\subset \{\beta\delta\epsilon, \alpha^2\beta, \gamma^3, \alpha^3\gamma, \alpha\gamma\delta\epsilon\}$. We have the AAD $\alpha\gamma\delta\epsilon = |^\beta\delta^\epsilon|^\delta\epsilon^\gamma|^\gamma\alpha^\beta|^\alpha\gamma^\epsilon|$ or $|^\beta\delta^\epsilon|^\delta\epsilon^\gamma|^\gamma\alpha^\beta|^\epsilon\gamma^\alpha|$. This gives a vertex $\alpha\beta\cdots = |^\beta\alpha^\gamma|\beta|\alpha|$. This implies that $\beta^2\cdots$ or $\beta\gamma\cdots$ is a vertex, contradicting the AVC.

Therefore, we know $\delta\epsilon\cdots = \beta\delta\epsilon$, a contradiction. Then $\#\beta \geq \#\beta\delta\epsilon + \#\alpha^2\beta > f$, a contradiction.

Case. $\{\beta\delta\epsilon, \alpha^2\beta, \alpha\gamma^2\}$

The angle sums of $\beta\delta\epsilon, \alpha^2\beta, \alpha\gamma^2$ and the angle sum for the pentagon imply

$$\alpha = \tfrac{8}{f},\ \beta = 2 - \tfrac{16}{f},\ \gamma = 1 - \tfrac{4}{f},\ \delta + \epsilon = \tfrac{16}{f}.$$

By $\beta \neq \gamma$, we have $f \neq 12$. By the angle values and $n_4 = n_5$, we get the

$$\text{AVC} \subset \{\beta\delta\epsilon, \alpha^2\beta, \alpha\gamma^2, \alpha^{\frac{f+4}{8}}\gamma, \alpha^{\frac{f}{4}}, \alpha^x\delta^{\frac{f-4x}{8}}\epsilon^{\frac{f-4x}{8}}, \alpha^y\gamma\delta^{\frac{f-8y+4}{16}}\epsilon^{\frac{f-8y+4}{16}}\},$$

where $x = 0, 1, 2, \cdots, \lfloor\frac{f}{4}\rfloor$, $y = 0, 1, 2, \cdots, \lfloor\frac{f+4}{8}\rfloor$. Then we know there is no $\beta^2\cdots$ and $\beta\gamma\cdots$ vertex, which further implies that (by AAD) the AVC $\subset \{\beta\delta\epsilon, \alpha^2\beta, \alpha\gamma^2, \alpha\gamma\delta\epsilon, \alpha^{\frac{f+4}{8}}\gamma, \alpha^{\frac{f}{4}}\}$.

Table 12: $\beta\delta\epsilon$, $\alpha^2\beta$ are vertices.

Vertex		Angles	Irrational Angle Lemma	Contradiction
$\beta\delta\epsilon, \alpha^2\beta$	$\alpha\beta\gamma$	$(\frac{1}{2}+\frac{2}{f}, 1-\frac{4}{f}, \frac{1}{2}+\frac{2}{f}, \delta, 1-\delta+\frac{4}{f})$	$n_4 = n_5$	No degree ≥ 4 vertex $\delta\epsilon\cdots$
	$\beta\gamma^2$			
	$\beta^2\gamma$	$(\frac{3}{5}+\frac{4}{5f}, \frac{4}{5}-\frac{8}{5f}, \frac{2}{5}+\frac{16}{5f}, \delta, \frac{6}{5}-\delta+\frac{8}{5f})$		
	$\alpha\gamma^3$	$(\frac{1}{2}+\frac{6}{f}, 1-\frac{12}{f}, \frac{1}{2}-\frac{2}{f}, \delta, 1-\delta+\frac{12}{f})$		
	$\beta\gamma^3$	$(\frac{3}{5}+\frac{12}{5f}, \frac{4}{5}-\frac{24}{5f}, \frac{2}{5}+\frac{8}{5f}, \delta, \frac{6}{5}-\delta+\frac{24}{5f})$		
	γ^4	$(\frac{1}{2}+\frac{4}{f}, 1-\frac{8}{f}, \frac{1}{2}, \delta, 1-\delta+\frac{8}{f})$		
	γ^5	$(\frac{3}{5}+\frac{4}{f}, \frac{4}{5}-\frac{8}{f}, \frac{2}{5}, \delta, \frac{6}{5}-\delta+\frac{8}{f})$		
	γ^3	$(\frac{1}{3}+\frac{4}{f}, \frac{4}{3}-\frac{8}{f}, \frac{2}{3}, \delta, \frac{2}{3}-\delta+\frac{8}{f})$		
	$\alpha\gamma^2$	$(\frac{8}{f}, 2-\frac{16}{f}, 1-\frac{4}{f}, \delta, -\delta+\frac{16}{f})$		

If $\alpha\gamma\delta\epsilon$ is a vertex, we have the unique AAD $\alpha\gamma\delta\epsilon = |^\gamma\alpha^\beta|^\alpha\gamma^\epsilon|^\beta\delta^\epsilon|^\delta\epsilon^\gamma|$. This gives a vertex $\alpha\beta\cdots = |^\gamma\alpha^\beta|^\gamma\alpha^\beta|^\delta\beta^\alpha|$. This gives a vertex $\beta\gamma\cdots$, contradicting the AVC. Therefore, we know $\delta\epsilon\cdots = \beta\delta\epsilon$, a contradiction.

The other cases lead to contradictions in a similar and much easier way. □

Lemma 17. *In a non-symmetric a^4b-tiling with general angles, if $\beta\delta\epsilon$ is the unique a^2b-vertex type, then α^3, $\alpha^2\gamma$, $\alpha\beta\gamma$ or $\alpha\gamma^2$ must appear. In each case, three linearly independent vertices in Table 13 must appear.*

Proof. By Proposition 4, we know that $\alpha^2\beta$ is not a vertex. We divide the discussion into five cases in Figure 26, according to the location of the vertex H of a special tile. Our strategy is to identify three linearly independent vertices.

Case $H = \alpha_1\cdots$

Let the first of Figure 26 be the center tile T_1 in the partial neighborhood in Figure 27. Since $\beta\delta\epsilon$ is the unique a^2b-vertex type and $H = \alpha_1\cdots$, the degree 3 vertex $\epsilon_1\cdots = \beta\delta\epsilon$ determines T_3, T_4. The two pictures show two possible arrangements of T_5. In the right-hand side of Figure 27, we have the degree 3 vertex $\beta_1\delta_5\cdots = \beta_1\delta_5\epsilon_6$, which determines T_6.

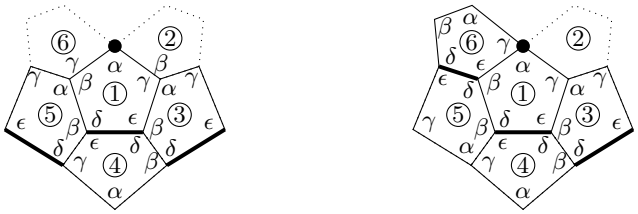


Figure 27: $H = \alpha_1\cdots$.

In the left-hand side of Figure 27, we have $\alpha_5\beta_1\cdots = \alpha\beta^2$ or $\alpha\beta\gamma$, $\alpha_3\gamma_1\cdots = \alpha^2\gamma$, $\alpha\beta\gamma$ or $\alpha\gamma^2$. By $\beta \neq \gamma$, we get $(\alpha_5\beta_1\cdots, \alpha_3\gamma_1\cdots) = (\alpha\beta^2, \alpha^2\gamma), (\alpha\beta\gamma, \alpha^2\gamma)$

or $(\alpha\beta\gamma, \alpha\beta\gamma)$. If $\alpha_5\beta_1\cdots = \alpha_3\gamma_1\cdots = \alpha\beta\gamma$, then we get

$$H = \{\alpha^3, \alpha^4, \alpha^3\beta, \alpha^3\gamma, \alpha^2\delta^2, \alpha^2\delta\epsilon, \alpha^2\epsilon^2, \alpha^5, \alpha^4\beta, \alpha^4\gamma, \alpha^3\beta^2, \alpha^3\delta^2, \alpha^3\delta\epsilon, \alpha^3\epsilon^2, \\ \alpha^2\beta\delta^2, \alpha^2\beta\epsilon^2, \alpha^2\gamma\delta^2, \alpha^2\gamma\epsilon^2, \alpha\delta^3\epsilon, \alpha\delta\epsilon^3\}.$$

In the right-hand side of Figure 27, we have $\alpha_3\gamma_1\cdots = \alpha_2\alpha_3\gamma_1$, $\alpha_3\beta_2\gamma_1$ or $\alpha_3\gamma_1\gamma_2$. Each case induces the following H respectively:

$$H = \{\alpha\beta\gamma, \alpha\gamma^2, \alpha\beta^2\gamma, \alpha\beta\gamma^2, \alpha\gamma^3, \alpha\beta^3\gamma, \alpha\beta^2\gamma^2, \alpha\beta\gamma^3, \alpha\beta\gamma\delta^2, \alpha\beta\gamma\epsilon^2, \alpha\gamma^4, \alpha\gamma^2\delta^2, \\ \alpha\gamma^2\delta\epsilon, \alpha\gamma^2\epsilon^2\}$$
$$\text{or } \{\alpha^2\gamma, \alpha^3\gamma, \alpha\gamma\delta^2, \alpha^4\gamma, \alpha^2\gamma\delta^2, \alpha^2\gamma\epsilon^2, \alpha\gamma^2\delta^2\}$$
$$\text{or } \{\alpha^2\gamma, \alpha^3\gamma, \alpha^2\beta\gamma, \alpha\gamma\delta\epsilon, \alpha\gamma\epsilon^2, \alpha^4\gamma, \alpha^3\beta\gamma, \alpha^2\beta^2\gamma, \alpha^2\gamma\delta^2, \alpha^2\gamma\delta\epsilon, \alpha^2\gamma\epsilon^2\}.$$

Case $H = \beta_1\cdots$

Let the second of Figure 26 be the center tile T_1 in the partial neighborhood in Figure 28. Since $\beta\delta\epsilon$ is an a^2b-vertex and $H = \beta_1\cdots$, we have the degree 3 vertex $\epsilon_1\cdots = \beta\delta\epsilon$, which determines T_4, T_5. The four pictures show four possible arrangements of T_3.

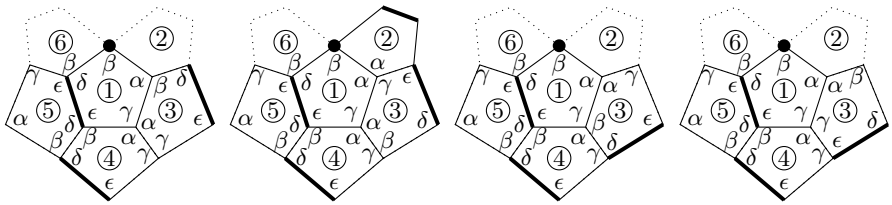


Figure 28: $H = \beta_1\cdots$.

In the first picture, we have $\alpha_1\beta_3\cdots = \alpha\beta^2$ or $\alpha\beta\gamma$. In the second picture, we have $\alpha_1\gamma_3\cdots = \alpha^2\gamma$, $\alpha\beta\gamma$ or $\alpha\gamma^2$. If $\alpha_1\gamma_3\cdots = \alpha^2\gamma$, we have

$$H = \{\alpha\beta^2, \alpha\beta\gamma, \alpha^2\beta^2, \alpha\beta^3, \alpha\beta^2\gamma, \alpha\beta\gamma^2, \beta^2\delta^2, \beta\gamma\delta^2, \alpha^2\beta^3, \alpha\beta^4, \alpha\beta^3\gamma, \alpha\beta^2\gamma^2, \\ \alpha\beta^2\delta^2, \alpha\beta^2\epsilon^2, \alpha\beta\gamma^3, \alpha\beta\gamma\delta^2, \alpha\beta\gamma\epsilon^2, \beta^3\delta^2, \beta^2\gamma\delta^2, \beta\gamma^2\delta^2\}.$$

In the third and fourth pictures, we have $\alpha_1\alpha_3\cdots = \alpha^3$ or $\alpha^2\gamma$.

Case $H = \gamma_1\cdots$

Let the fifth of Figure 26 be the center tile T_1 in the partial neighborhood in Figure 29. Since $\beta\delta\epsilon$ is an a^2b-vertex and $H = \gamma_1\cdots$, we have the degree 3 vertex $\epsilon_1\cdots = \beta\delta\epsilon$, which determines T_3. The two pictures show two possible arrangements of T_4. In the first picture of Figure 29, we have the degree 3 vertex $\alpha_4\beta_1\cdots = \alpha_4\beta_1\beta_5$ or $\alpha_4\beta_1\gamma_5$, which determines α_5. In the second picture of Figure 29, we have the degree 3 vertex $\beta_1\delta_4\cdots = \beta\delta\epsilon$, which determines T_5.

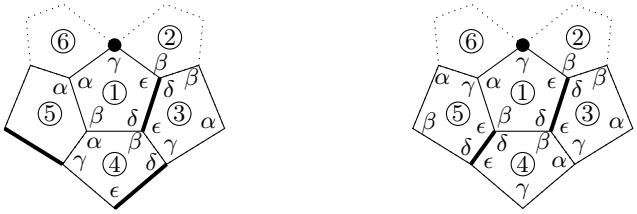


Figure 29: $H = \gamma_1 \cdots$.

In the first picture of Figure 29, we have $\alpha_1\alpha_5 \cdots = \alpha^3$ or $\alpha^2\gamma$. In the second picture of Figure 29, we have $\alpha_1\gamma_5 \cdots = \alpha^2\gamma$, $\alpha\beta\gamma$ or $\alpha\gamma^2$. Each case induces the following H respectively:

$$\begin{aligned} H = \{&\alpha\beta\gamma, \alpha\gamma^2, \alpha\beta^2\gamma, \alpha\beta\gamma^2, \alpha\gamma^3, \beta\gamma\delta^2, \gamma^2\delta^2, \gamma^2\delta\epsilon, \alpha\beta^3\gamma, \alpha\beta^2\gamma^2, \alpha\beta\gamma^3, \alpha\beta\gamma\delta^2, \\ &\alpha\beta\gamma\epsilon^2, \alpha\gamma^4, \alpha\gamma^2\delta^2, \alpha\gamma^2\delta\epsilon, \alpha\gamma^2\epsilon^2, \beta^2\gamma\delta^2, \beta\gamma^2\delta^2, \gamma^3\delta^2, \gamma^3\delta\epsilon\} \\ \text{or } &\{\alpha^2\gamma, \alpha^3\gamma, \alpha\gamma\delta^2, \alpha^4\gamma, \alpha^2\gamma\delta^2, \alpha^2\gamma\epsilon^2, \alpha\gamma^2\delta^2, \gamma\delta^4, \gamma\delta^3\epsilon\} \\ \text{or } &\{\alpha^2\gamma, \alpha^3\gamma, \alpha^2\beta\gamma, \alpha\gamma\delta^2, \alpha\gamma\delta\epsilon, \alpha\gamma\epsilon^2, \alpha^4\gamma, \alpha^3\beta\gamma, \alpha^2\beta^2\gamma, \alpha^2\gamma\delta^2, \alpha^2\gamma\delta\epsilon, \alpha^2\gamma\epsilon^2, \\ &\alpha\beta\gamma\epsilon^2, \gamma\delta^3\epsilon, \gamma\delta^2\epsilon^2, \gamma\delta\epsilon^3\}. \end{aligned}$$

Case $H = \delta_1 \cdots$

Let the third of Figure 26 be the center tile T_1 in the partial neighborhood in Figure 30. Since $\beta\delta\epsilon$ is an a^2b-vertex and $H = \delta_1 \cdots$, we have the degree 3 vertex $\epsilon_1 \cdots = \beta\delta\epsilon$, which determines T_5 and T_6. The three pictures show three possible arrangements of T_4.

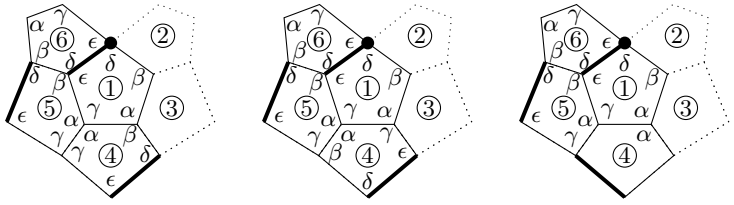


Figure 30: $H = \delta_1 \cdots$.

In the first picture, we have $\alpha_1\beta_4 \cdots = \alpha\beta^2, \alpha\beta\gamma$. In the second picture, we have $\alpha_1\gamma_3 \cdots = \alpha^2\gamma$, $\alpha\beta\gamma$ or $\alpha\gamma^2$. If $\alpha_1\gamma_4 \cdots = \alpha^2\gamma$, then $\beta_1 \cdots = \alpha\beta^2, \alpha\beta\gamma, \beta^3, \beta^2\gamma, \beta\gamma^2$. In the third picture, we have $\alpha_5\gamma_1 \cdots = \alpha\beta\gamma$ or $\alpha\gamma^2$, $\alpha_1\alpha_4 \cdots = \alpha^3$ or $\alpha^2\gamma$.

Case $H = \epsilon_1 \cdots$

Let the fourth of Figure 26 be the center tile T_1 in the partial neighborhood in Figure 31. Since $\beta\delta\epsilon$ is an a^2b-vertex and $H = \epsilon_1 \cdots$, we have the degree 3

vertex $\delta_1 \cdots = \beta\delta\epsilon$, which determines T_2. The four pictures show two possible arrangements of T_3. In the second, third and fourth pictures, we have the degree 3 vertex $\beta_1\delta_3 \cdots = \beta\delta\epsilon$, which determines T_4. In the second picture, we have the degree 3 vertex $\alpha_1\gamma_4 \cdots = \alpha_1\alpha_5\gamma_4$, which determines α_5. In the third picture, we have the degree 3 vertex $\alpha_1\gamma_4 \cdots = \alpha_1\beta_5\gamma_4$, which determines T_5. In the fourth picture, we have the degree 3 vertex $\alpha_1\gamma_4 \cdots = \alpha_1\gamma_4\gamma_5$, which determines T_5.

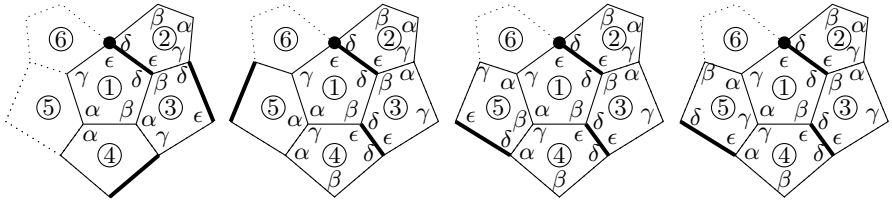


Figure 31: $H = \epsilon_1 \cdots$.

In the first picture, we have $\alpha_3\beta_1 \cdots = \alpha\beta^2$ or $\alpha\beta\gamma$, $\alpha_1\alpha_4 \cdots = \alpha^3$ or $\alpha^2\gamma$.

In the second picture, we have $\gamma_1 \cdots = \alpha\beta\gamma$, $\alpha\gamma^2$, $\beta^2\gamma$, $\beta\gamma^2$ or γ^3.

In the third picture, we have $\alpha_5\gamma_1 \cdots = \alpha^2\gamma$ or $\alpha\beta\gamma$. If $\alpha_5\gamma_1 \cdots = \alpha\beta\gamma$, we have

$$H = \{\alpha^2\delta\epsilon, \delta^3\epsilon, \alpha^3\delta\epsilon, \alpha\delta^3\epsilon, \alpha\delta\epsilon^3, \gamma\delta^3\epsilon\}.$$

In the fourth picture, we have $\alpha_5\gamma_1 \cdots = \alpha^2\gamma$, $\alpha\beta\gamma$ or $\alpha\gamma^2$. If $\alpha_5\gamma_1 \cdots = \alpha\gamma^2$, we have

$$H = \{\alpha^2\delta\epsilon, \alpha\gamma\delta\epsilon, \delta^2\epsilon^2, \delta\epsilon^3, \alpha^3\delta\epsilon, \alpha^2\gamma\delta\epsilon, \alpha\gamma^2\delta\epsilon, \alpha\delta^2\epsilon^2, \alpha\delta^3\epsilon, \alpha\delta\epsilon^3, \gamma\delta^2\epsilon^2, \gamma\delta\epsilon^3\}.$$

By the previous discussion, we get Table 13.

□

Proposition 5. *There is no non-symmetric a^4b-tiling with general angles, such that $\beta\delta\epsilon$ is the unique a^2b-vertex type and $\alpha^2\gamma$ is a vertex.*

Proof. By Lemma 17, one of the vertices in the last column of Table 13: $\{\alpha\beta^2$, $\alpha\beta\gamma$, $\alpha\gamma^2$, β^3, $\beta^2\gamma$, $\beta\gamma^2$, γ^3, $\alpha^2\beta^2$, $\alpha\beta^3$, $\alpha\beta^2\gamma$, $\alpha\beta\gamma^2$, $\alpha\gamma^3$, $\beta^2\delta^2$, $\beta\gamma\delta^2$, $\gamma^2\delta^2$, $\gamma^2\delta\epsilon$, $\alpha^2\beta^3$, $\alpha\beta^4$, $\alpha\beta^3\gamma$, $\alpha\beta^2\gamma^2$, $\alpha\beta^2\delta^2$, $\alpha\beta^2\epsilon^2$, $\alpha\beta\gamma^3$, $\alpha\beta\gamma\delta^2$, $\alpha\beta\gamma\epsilon^2$, $\alpha\gamma^4$, $\alpha\gamma^2\delta^2$, $\alpha\gamma^2\delta\epsilon$, $\alpha\gamma^2\epsilon^2$, $\beta^3\delta^2$, $\beta^2\gamma\delta^2$, $\beta\gamma^2\delta^2$, $\gamma^3\delta^2$, $\gamma^3\delta\epsilon\}$ must appear.

For each combination of 3 vertices, we will apply the irrational angle lemma and AAD to get Table 14. The proof is completed by Table 14, 15, 16, 17, 18, 19 and 20.

Case. $\{\beta\delta\epsilon, \alpha^2\gamma, \beta\gamma^2\}$

By the irrational angle lemma, we have $n_4 = n_5$. The angle sums of $\beta\delta\epsilon, \alpha^2\gamma, \beta\gamma^2$ and the angle sum for the pentagon imply

$$\alpha = 1 - \tfrac{4}{f},\ \beta = 2 - \tfrac{16}{f},\ \gamma = \tfrac{8}{f},\ \delta + \epsilon = \tfrac{16}{f}.$$

Table 13: $\beta\delta\epsilon$ as the unique a^2b-vertex type.

a^2b-Vertex	H	a^3-Vertex	Vertex
$\beta\delta\epsilon$	α	$\alpha^2\gamma$	$\alpha\beta^2, \alpha\beta\gamma, \alpha\gamma^2, \alpha\beta^2\gamma, \alpha\beta\gamma^2, \alpha\gamma^3, \alpha\beta^3\gamma, \alpha\beta^2\gamma^2, \alpha\beta\gamma^3, \alpha\beta\gamma\delta^2, \alpha\beta\gamma\epsilon^2,$ $\alpha\gamma^4, \alpha\gamma^2\delta^2, \alpha\gamma^2\delta\epsilon, \alpha\gamma^2\epsilon^2$
		$\alpha\beta\gamma$	$\alpha^3, \alpha^4, \alpha^3\beta, \alpha^3\gamma, \alpha^2\delta^2, \alpha^2\delta\epsilon, \alpha^2\epsilon^2, \alpha\gamma\delta^2, \alpha^5, \alpha^4\beta, \alpha^4\gamma, \alpha^3\beta^2, \alpha^3\delta^2,$ $\alpha^3\delta\epsilon, \alpha^3\epsilon^2, \alpha^2\beta\delta^2, \alpha^2\beta\epsilon^2, \alpha^2\gamma\delta^2, \alpha^2\gamma\epsilon^2, \alpha\gamma^2\delta^2, \alpha\delta^3\epsilon, \alpha\delta\epsilon^3$
		$\alpha\gamma^2$	$\alpha^3\gamma, \alpha^2\beta\gamma, \alpha\gamma\delta\epsilon, \alpha\gamma\epsilon^2, \alpha^4\gamma, \alpha^3\beta\gamma, \alpha^2\beta^2\gamma, \alpha^2\gamma\delta^2, \alpha^2\gamma\delta\epsilon, \alpha^2\gamma\epsilon^2$
	β	$\alpha^2\gamma$	$\alpha\beta^2, \alpha\beta\gamma, \alpha\gamma^2, \alpha^2\beta^2, \alpha\beta^3, \alpha\beta^2\gamma, \alpha\beta\gamma^2, \beta^2\delta^2, \beta\gamma\delta^2, \alpha^2\beta^3, \alpha\beta^4, \alpha\beta^3\gamma,$ $\alpha\beta^2\gamma^2, \alpha\beta\gamma^3, \alpha\beta^2\delta^2, \alpha\beta^2\epsilon^2, \alpha\beta\gamma\delta^2, \alpha\beta\gamma\epsilon^2, \beta^3\delta^2, \beta^2\gamma\delta^2, \beta\gamma^2\delta^2$
		$\alpha\beta\gamma$	α^3
		$\alpha\gamma^2$	α^3
	γ	α^3	$\alpha\beta^2$
		$\alpha^2\gamma$	$\alpha\beta^2, \alpha\beta\gamma, \alpha\gamma^2, \alpha\beta^2\gamma, \alpha\beta\gamma^2, \alpha\gamma^3, \beta\gamma\delta^2, \gamma^2\delta^2, \gamma^2\delta\epsilon, \alpha\beta^3\gamma, \alpha\beta^2\gamma^2, \alpha\beta\gamma^3,$ $\alpha\beta\gamma\delta^2, \alpha\beta\gamma\epsilon^2, \alpha\gamma^4, \alpha\gamma^2\delta^2, \alpha\gamma^2\delta\epsilon, \alpha\gamma^2\epsilon^2, \beta^2\gamma\delta^2, \beta\gamma^2\delta^2, \gamma^3\delta^2, \gamma^3\delta\epsilon$
		$\alpha\beta\gamma$	$\alpha^3, \alpha^3\gamma, \alpha\gamma\delta^2, \alpha^4\gamma, \alpha^2\gamma\delta^2, \alpha^2\gamma\epsilon^2, \alpha\gamma^2\delta^2, \gamma\delta^4, \gamma\delta^3\epsilon$
		$\alpha\gamma^2$	$\alpha^3\gamma, \alpha^2\beta\gamma, \alpha\gamma\delta\epsilon, \alpha\gamma\delta^2, \alpha\gamma\epsilon^2, \alpha^4\gamma, \alpha^3\beta\gamma, \alpha^2\beta^2\gamma, \alpha^2\gamma\delta^2, \alpha^2\gamma\delta\epsilon, \alpha^2\gamma\epsilon^2,$ $\alpha\beta\gamma\epsilon^2, \gamma\delta^3\epsilon, \gamma\delta^2\epsilon^2, \gamma\delta\epsilon^3$
	δ	$\alpha^2\gamma$	$\alpha\beta^2, \alpha\beta\gamma, \alpha\gamma^2, \beta^3, \beta^2\gamma, \beta\gamma^2$
		$\alpha\beta\gamma$	α^3
		$\alpha\gamma^2$	α^3
	ϵ	α^3	$\alpha\beta^2$
		$\alpha^2\gamma$	$\alpha\beta^2, \alpha\beta\gamma, \alpha\gamma^2, \beta^2\gamma, \beta\gamma^2, \gamma^3$
		$\alpha\beta\gamma$	$\alpha^3, \alpha^2\delta\epsilon, \delta^3\epsilon, \alpha^3\delta\epsilon, \alpha\delta^3\epsilon, \alpha\delta\epsilon^3, \gamma\delta^3\epsilon$
		$\alpha\gamma^2$	$\alpha^2\delta\epsilon, \alpha\gamma\delta\epsilon, \delta^2\epsilon^2, \delta\epsilon^3, \alpha^3\delta\epsilon, \alpha^2\gamma\delta\epsilon, \alpha\gamma^2\delta\epsilon, \alpha\delta^2\epsilon^2, \alpha\delta^3\epsilon, \alpha\delta\epsilon^3, \gamma\delta^2\epsilon^2, \gamma\delta\epsilon^3$

Table 14: $\beta\delta\epsilon$, $\alpha^2\gamma$ are vertices.

Vertex		Angles	Irrational Angle Lemma	AAD	Contradiction
$\beta\delta\epsilon, \alpha^2\gamma$	$\alpha\beta^2\epsilon^2$	$(1-\frac{4}{f}, \frac{1}{2}-\epsilon+\frac{2}{f}, \frac{8}{f}, \frac{3}{2}-\frac{2}{f}, \epsilon)$	$n_2 = n_5$	$\lvert\gamma\epsilon^\delta\rvert\epsilon^\delta\epsilon^\gamma\rvert\ldots$	No $\delta^2\cdots$ by the irrational angle lemma
	$\alpha\beta\gamma\epsilon^2$	$(1-\frac{4}{f}, 3-2\delta+\frac{4}{f}, \frac{8}{f}, \delta, -1+\delta-\frac{4}{f})$	$2n_2 = n_4 + n_5$		
	$\alpha\gamma^2\epsilon^2$	$(1-\frac{4}{f}, \frac{3}{2}-\delta+\frac{6}{f}, \frac{8}{f}, \delta, \frac{1}{2}-\frac{6}{f})$	$n_2 = n_4$	$\lvert\beta\delta^\epsilon\rvert\epsilon\delta^\beta\rvert\ldots$	No $\epsilon^2\cdots$
	$\beta^2\delta^2$	$(1-\frac{4}{f}, 1-\delta, \frac{8}{f}, \delta, 1)$			
	$\alpha\beta^2\delta^2$	$(1-\frac{4}{f}, \frac{1}{2}-\delta+\frac{2}{f}, \frac{8}{f}, \delta, \frac{3}{2}-\frac{2}{f})$			
	$\beta^2\gamma\delta^2$	$(1-\frac{4}{f}, 1-\delta-\frac{4}{f}, \frac{8}{f}, \delta, 1+\frac{4}{f})$			
	$\beta\gamma\delta^2$	$(1-\frac{4}{f}, 2-2\delta-\frac{8}{f}, \frac{8}{f}, \delta, \delta+\frac{8}{f})$	$2n_2 = n_4 + n_5$		
	$\alpha\beta\gamma\delta^2$	$(1-\frac{4}{f}, 1-2\delta-\frac{4}{f}, \frac{8}{f}, \delta, 1+\delta+\frac{4}{f})$			
	$\beta\gamma^2\delta^2$	$(1-\frac{4}{f}, 2-2\delta-\frac{16}{f}, \frac{8}{f}, \delta, \delta+\frac{16}{f})$			
	$\gamma^2\delta^2$	$(1-\frac{4}{f}, 1-\epsilon+\frac{8}{f}, \frac{8}{f}, 1-\frac{8}{f}, \epsilon)$	$n_2 = n_5$		
	$\alpha\gamma^2\delta^2$	$(1-\frac{4}{f}, \frac{3}{2}-\epsilon+\frac{6}{f}, \frac{8}{f}, \frac{1}{2}-\frac{6}{f}, \epsilon)$			
	$\gamma^3\delta^2$	$(1-\frac{4}{f}, 1-\epsilon+\frac{12}{f}, \frac{8}{f}, 1-\frac{12}{f}, \epsilon)$			
	$\beta^3\delta^2$	$(1-\frac{4}{f}, \frac{2}{3}-\frac{2}{3}\delta, \frac{8}{f}, \delta, \frac{4}{3}-\frac{1}{3}\delta)$	$2n_2 + n_5 = 3n_4$		
	$\alpha^2\beta^2$	$(1-\frac{4}{f}, \frac{4}{f}, \frac{8}{f}, \delta, 2-\delta-\frac{4}{f})$	$n_4 = n_5$		No degree ≥ 4 vertex $\delta\epsilon\cdots$ (Similar to Proposition 4)
	$\alpha^2\beta^3$	$(1-\frac{4}{f}, \frac{8}{3f}, \frac{8}{f}, \delta, 2-\delta-\frac{8}{3f})$			
	$\alpha\beta^2$	$(1-\frac{4}{f}, \frac{1}{2}+\frac{2}{f}, \frac{8}{f}, \delta, \frac{3}{2}-\delta-\frac{2}{f})$			See Table 15
	$\alpha\beta\gamma$	$(1-\frac{4}{f}, 1-\frac{4}{f}, \frac{8}{f}, \delta, 1-\delta+\frac{4}{f})$			
	$\beta^2\gamma$				
	β^3	$(1-\frac{4}{f}, \frac{2}{3}, \frac{8}{f}, \delta, \frac{4}{3}-\delta)$			
	$\alpha\beta^3$	$(1-\frac{4}{f}, \frac{1}{3}+\frac{4}{3f}, \frac{8}{f}, \delta, \frac{5}{3}-\delta-\frac{4}{3f})$			See Table 16
	$\alpha\beta\gamma^2$	$(1-\frac{4}{f}, 1-\frac{12}{f}, \frac{8}{f}, \delta, 1-\delta+\frac{12}{f})$			
	$\alpha\beta^2\gamma$	$(1-\frac{4}{f}, \frac{1}{2}-\frac{2}{f}, \frac{8}{f}, \delta, \frac{3}{2}-\delta+\frac{2}{f})$			See Table 17
	$\alpha\beta^3\gamma$	$(1-\frac{4}{f}, \frac{1}{3}-\frac{4}{3f}, \frac{8}{f}, \delta, \frac{5}{3}-\delta+\frac{4}{3f})$			
	$\alpha\beta^4$	$(1-\frac{4}{f}, \frac{1}{4}+\frac{1}{f}, \frac{8}{f}, \delta, \frac{7}{4}-\delta-\frac{1}{f})$			See Table 18
	$\alpha\beta^2\gamma^2$	$(1-\frac{4}{f}, \frac{1}{2}-\frac{6}{f}, \frac{8}{f}, \delta, \frac{3}{2}-\delta+\frac{6}{f})$			
	$\alpha\beta\gamma^3$	$(1-\frac{4}{f}, 1-\frac{20}{f}, \frac{8}{f}, \delta, 1-\delta+\frac{20}{f})$			See Table 19
	$\alpha\gamma^2\delta\epsilon$	$(1-\frac{4}{f}, 1+\frac{12}{f}, \frac{8}{f}, \delta, 1-\delta-\frac{12}{f})$			
	$\gamma^2\delta\epsilon$	$(1-\frac{4}{f}, \frac{10}{f}, \frac{8}{f}, \delta, 2-\delta-\frac{10}{f})$			See Table 20
	$\gamma^3\delta\epsilon$	$(1-\frac{4}{f}, \frac{24}{f}, \frac{8}{f}, \delta, 2-\delta-\frac{24}{f})$			
	$\beta\gamma^2$	$(1-\frac{4}{f}, 2-\frac{16}{f}, \frac{8}{f}, \delta, -\delta+\frac{16}{f})$			See page 32
	$\alpha\gamma^2$	$(\frac{2}{3}, 2-\delta-\epsilon, \frac{2}{3}, \delta, \epsilon), f = 12$			See page 35
	γ^3				
	$\alpha\gamma^3$	$(\frac{4}{5}, 2-\delta-\epsilon, \frac{2}{5}, \delta, \epsilon), f = 20$			
	$\alpha\gamma^4$	$(\frac{6}{7}, 2-\delta-\epsilon, \frac{2}{7}, \delta, \epsilon), f = 28$			

Table 15: AVC for Cases $\{\beta\delta\epsilon, \alpha^2\gamma, \alpha\beta^2\}$, $\{\beta\delta\epsilon, \alpha^2\gamma, \alpha\beta\gamma(\beta^2\gamma)\}$ and $\{\beta\delta\epsilon, \alpha^2\gamma, \beta^3\}$.

AVC for $\{\beta\delta\epsilon, \alpha^2\gamma, \alpha\beta^2\}$		AVC for $\{\beta\delta\epsilon, \alpha^2\gamma, \alpha\beta\gamma(\beta^2\gamma)\}$		AVC for $\{\beta\delta\epsilon, \alpha^2\gamma, \beta^3\}$	
f	vertex	f	vertex	f	vertex
all $16s+12, s=1,2,\cdots$	$\beta\delta\epsilon, \alpha^2\gamma, \alpha\beta^2$ $\alpha\beta\gamma^{s+1}, \alpha\gamma^{2s+2}, \beta^3\gamma^s,$ $\beta^2\gamma^{2s+1}, \beta\gamma^{3s+2}, \gamma^{s+1}\delta\epsilon,$ γ^{4s+3}	all $8s+4, s=2,3,\cdots$	$\beta\delta\epsilon, \alpha^2\gamma, \alpha\beta\gamma, \beta^2\gamma$ $\alpha\gamma^{s+1}, \beta\gamma^{s+1}, \gamma^s\delta\epsilon,$ γ^{2s+1}	all $24s, s=2,3,\cdots$ $24s+12, s=2,3,\cdots$	$\beta\delta\epsilon, \alpha^2\gamma, \beta^3$ $\beta^2\gamma^{2s}, \beta\gamma^{4s}, \gamma^{6s}, \gamma^{2s}\delta\epsilon$ $\alpha\beta\gamma^{s+1}, \alpha\gamma^{3s+2}, \beta^2\gamma^{2s+1},$ $\beta\gamma^{4s+2}, \gamma^{6s+3}, \gamma^{2s+1}\delta\epsilon$
By the balance lemma, $\gamma^k\delta\epsilon$ must appear. Its AAD gives $\alpha\vert\delta$, $\alpha\vert\epsilon$, $\delta\vert\delta$, $\delta\vert\epsilon$ or $\epsilon\vert\epsilon$, contradicting the AVC					

Table 16: AVC for Cases $\{\beta\delta\epsilon, \alpha^2\gamma, \alpha\beta^3\}$ and $\{\beta\delta\epsilon, \alpha^2\gamma, \alpha\beta\gamma^2\}$.

AVC for $\{\beta\delta\epsilon, \alpha^2\gamma, \alpha\beta^3\}$		AVC for $\{\beta\delta\epsilon, \alpha^2\gamma, \alpha\beta\gamma^2\}$	
f	vertex	f	vertex
all $24s+20, s=1,2,\cdots$	$\beta\delta\epsilon, \alpha^2\gamma, \alpha\beta^3$ $\alpha\beta^2\gamma^{s+1}, \alpha\beta\gamma^{2s+2}, \alpha\gamma^{3s+3}, \beta^5\gamma^s, \beta^4\gamma^{2s+1},$ $\beta^3\gamma^{3s+2}, \beta^2\gamma^{4s+3}, \beta\gamma^{5s+4}, \gamma^{6s+5}, \gamma^{s+1}\delta\epsilon$	all 28 36 $8s+4, s=5,6,\cdots$	$\beta\delta\epsilon, \alpha^2\gamma, \alpha\beta\gamma^2, \beta^2\gamma^3$ $\alpha\beta^2, \beta^3\gamma, \gamma^2\delta\epsilon, \alpha\gamma^4, \beta\gamma^5, \gamma^7$ $\beta^3, \gamma^3\delta\epsilon, \alpha\gamma^5, \beta\gamma^6, \gamma^9$ $\alpha\gamma^{s+1}, \beta\gamma^{s+2}, \gamma^{2s+1}, \gamma^{s-1}\delta\epsilon$
By the balance lemma, $\gamma^k\delta\epsilon$ must appear. Its AAD gives $\alpha\vert\delta$, $\alpha\vert\epsilon$, $\delta\vert\delta$, $\delta\vert\epsilon$ or $\epsilon\vert\epsilon$, contradicting the AVC			

Table 17: AVC for Cases $\{\beta\delta\epsilon, \alpha^2\gamma, \alpha\beta^2\gamma\}$ and $\{\beta\delta\epsilon, \alpha^2\gamma, \alpha\beta^3\gamma\}$.

AVC for $\{\beta\delta\epsilon, \alpha^2\gamma, \alpha\beta^2\gamma\}$		AVC for $\{\beta\delta\epsilon, \alpha^2\gamma, \alpha\beta^3\gamma\}$	
f	vertex	f	vertex
all $16s+4, s=2,3,\cdots$	$\beta\delta\epsilon, \alpha^2\gamma, \alpha\beta^2\gamma, \beta^4\gamma$ $\alpha\beta\gamma^{s+1}, \alpha\gamma^{2s+1}, \beta^3\gamma^{s+1}, \beta^2\gamma^{2s+1},$ $\beta\gamma^{3s+1}, \gamma^{4s+1}, \gamma^s\delta\epsilon$	all $24s+4, s=2,3,\cdots$	$\beta\delta\epsilon, \alpha^2\gamma, \alpha\beta^3\gamma, \beta^6\gamma$ $\alpha\beta^2\gamma^{s+1}, \alpha\beta\gamma^{2s+1}, \alpha\gamma^{3s+1}, \beta^5\gamma^{s+1}, \beta^4\gamma^{2s+1},$ $\beta^3\gamma^{3s+1}, \beta^2\gamma^{4s+1}, \beta\gamma^{5s+1}, \gamma^{6s+1}, \gamma^s\delta\epsilon$
By the balance lemma, $\gamma^k\delta\epsilon$ must appear. Its AAD gives $\alpha\vert\delta$, $\alpha\vert\epsilon$, $\delta\vert\epsilon$, $\delta\vert\delta$ or $\epsilon\vert\epsilon$, contradicting the AVC			

Table 18: AVC for Cases $\{\beta\delta\epsilon, \alpha^2\gamma, \alpha\beta^4\}$ and $\{\beta\delta\epsilon, \alpha^2\gamma, \alpha\beta^2\gamma^2\}$.

AVC for $\{\beta\delta\epsilon, \alpha^2\gamma, \alpha\beta^4\}$		AVC for $\{\beta\delta\epsilon, \alpha^2\gamma, \alpha\beta^2\gamma^2\}$	
f	vertex	f	vertex
all $32s+28, s=1,2,\cdots$	$\beta\delta\epsilon, \alpha^2\gamma, \alpha\beta^4$ $\alpha\beta^3\gamma^{s+1}, \alpha\beta^2\gamma^{2s+2}, \alpha\beta\gamma^{3s+3}, \alpha\gamma^{4s+4},$ $\beta^7\gamma^s, \beta^6\gamma^{2s+1}, \beta^5\gamma^{3s+2}, \beta^4\gamma^{4s+3},$ $\beta^3\gamma^{5s+4}, \beta^2\gamma^{6s+5}, \beta\gamma^{7s+6}, \gamma^{8s+7}, \gamma^{s+1}\delta\epsilon$	all 44 60 $16s+12, s=4,5,\cdots$	$\beta\delta\epsilon, \alpha^2\gamma, \alpha\beta^2\gamma^2, \beta^4\gamma^3$ $\alpha\beta^3, \gamma^2\delta\epsilon, \alpha\beta\gamma^4, \beta^5\gamma, \alpha\gamma^6, \beta^3\gamma^5, \beta^2\gamma^7, \beta\gamma^9, \gamma^{11}$ $\beta^5, \gamma^3\delta\epsilon, \alpha\beta\gamma^5, \alpha\gamma^8, \beta^3\gamma^6, \beta^2\gamma^9, \beta\gamma^{12}, \gamma^{15}$ $\alpha\beta\gamma^{s+2}, \alpha\gamma^{2s+2}, \beta^3\gamma^{s+3}, \beta^2\gamma^{2s+3}, \beta\gamma^{3s+3}, \gamma^{4s+3}, \gamma^s\delta\epsilon$
By the balance lemma, $\gamma^k\delta\epsilon$ must appear. Its AAD gives $\alpha\vert\delta$, $\alpha\vert\epsilon$, $\delta\vert\epsilon$, $\delta\vert\delta$ or $\epsilon\vert\epsilon$, contradicting the AVC			

Table 19: AVC for Cases $\{\beta\delta\epsilon, \alpha^2\gamma, \alpha\beta\gamma^3\}$ and $\{\beta\delta\epsilon, \alpha^2\gamma, \alpha\gamma^2\delta\epsilon\}$.

f	vertex	f	vertex
all	$\beta\delta\epsilon, \alpha^2\gamma, \alpha\beta\gamma^3, \beta^2\gamma^5$	all	$\beta\delta\epsilon, \alpha^2\gamma, \alpha\gamma^2\delta\epsilon, \gamma^3\delta^2\epsilon^2$
36	$\alpha\beta^2\gamma, \gamma^2\delta\epsilon, \alpha\gamma^5, \beta^4\gamma, \beta^3\gamma^3, \beta\gamma^7, \gamma^9$	28	$\beta\gamma^2, \alpha\gamma^4, \alpha\delta^2\epsilon^2, \gamma^5\delta\epsilon, \gamma\delta^3\epsilon^3, \gamma^7$
44	$\alpha\beta^2, \beta^3\gamma^2, \gamma^3\delta\epsilon, \alpha\gamma^6, \beta\gamma^8, \gamma^{11}$	36	$\beta\gamma^3, \alpha\gamma^5, \delta^3\epsilon^3, \gamma^6\delta\epsilon, \gamma^9$
52	$\beta^3\gamma, \gamma^4\delta\epsilon, \alpha\gamma^7, \beta\gamma^9, \gamma^{13}$	$8s+4, s=5,6,\cdots$	$\alpha\gamma^{s+1}, \beta\gamma^{s-1}, \gamma^{2s+1}, \gamma^{s+2}\delta\epsilon$
60	$\beta^3, \gamma^5\delta\epsilon, \alpha\gamma^8, \beta\gamma^{10}, \gamma^{15}$		
$8s+4, s=8,9\cdots$	$\alpha\gamma^{s+1}, \beta\gamma^{s+3}, \gamma^{2s+1}, \gamma^{s-2}\delta\epsilon$		
By the balance lemma, $\gamma^k\delta\epsilon$ must appear. Its AAD gives $\alpha\vert\delta$, $\alpha\vert\epsilon$, $\delta\vert\delta$, $\delta\vert\epsilon$ or $\epsilon\vert\epsilon$, contradicting the AVC		By the balance lemma, $\beta\gamma^k$ must appear. Its AAD gives $\alpha\vert\beta$, $\beta\vert\beta$ or $\delta\vert\epsilon$, contradicting the AVC	

Table 20: AVC for Cases $\{\beta\delta\epsilon, \alpha^2\gamma, \gamma^2\delta\epsilon\}$ and $\{\beta\delta\epsilon, \alpha^2\gamma, \gamma^3\delta\epsilon\}$.

AVC for $\{\beta\delta\epsilon, \alpha^2\gamma, \gamma^2\delta\epsilon\}$		AVC for $\{\beta\delta\epsilon, \alpha^2\gamma, \gamma^3\delta\epsilon\}$	
f	vertex	f	vertex
all	$\beta\delta\epsilon, \alpha^2\gamma, \gamma^2\delta\epsilon$	all	$\beta\delta\epsilon, \alpha^2\gamma, \gamma^3\delta\epsilon$
16	$\beta\gamma^2, \gamma^4, \delta^2\epsilon^2$	20	$\beta\gamma^2, \alpha\gamma^3, \alpha\gamma\delta\epsilon, \gamma^5, \gamma\delta^2\epsilon^2$
20	$\alpha\beta\gamma, \beta^2\gamma, \alpha\gamma^3, \beta\gamma^3, \gamma^5$	24	$\beta\gamma^3, \delta^2\epsilon^2, \gamma^6$
$8s, s=3,4,\cdots$	$\beta^x\gamma^{2s-2x}(x=0,1,\cdots,s)$	28	$\alpha\beta\gamma, \beta^2\gamma, \alpha\gamma^4, \beta\gamma^4, \gamma^7$
$8s+4, s=3,4,\cdots$	$\beta^x\gamma^{2s+1-2x}(x=0,1,\cdots,s),$ $\alpha\beta^y\gamma^{s+1-2y}(y=0,1,\cdots,\lfloor\frac{s-1}{2}\rfloor)$	$8s, s=4,5,\cdots$ $8s+4, s=4,5,\cdots$	$\beta^x\gamma^{2s-3x}(x=0,1,\cdots,\lfloor\frac{2s}{3}\rfloor)$ $\beta^x\gamma^{2s+1-3x}(x=0,1,\cdots,\lfloor\frac{2s+1}{3}\rfloor),$ $\alpha\beta^y\gamma^{s+1-3y}(y=0,1,\cdots,\lfloor\frac{s+1}{3}\rfloor)$
By the balance lemma, $\beta^k\gamma^l$ or $\alpha\beta^k\gamma^l$ must appear. Its AAD gives $\alpha\vert\beta$, $\beta\vert\beta$, $\alpha\vert\delta$, $\alpha\vert\epsilon$, $\delta\vert\delta$, $\delta\vert\epsilon$ or $\epsilon\vert\epsilon$, contradicting the AVC			

By the angle values, we get the

$$\text{AVC} \subset \{\beta\delta\epsilon, \alpha^2\gamma, \alpha\beta\gamma, \beta\gamma^2, \alpha\gamma^{\frac{f+4}{8}}, \alpha\gamma^x\delta^{\frac{f-8x+4}{16}}\epsilon^{\frac{f-8x+4}{16}}, \gamma^y\delta^{\frac{f-4y}{8}}\epsilon^{\frac{f-4y}{8}}\},$$

where $x = 0, 1, 2, \cdots, \lfloor\frac{f+4}{8}\rfloor$, $y = 0, 1, 2, \cdots, \lfloor\frac{f}{4}\rfloor$. If $f \neq 4k (k \geq 3)$, we know $\alpha\gamma^x\delta^{\frac{f-8x+4}{16}}\epsilon^{\frac{f-8x+4}{16}}, \gamma^y\delta^{\frac{f-4y}{8}}\epsilon^{\frac{f-4y}{8}}$ are not vertices. Then $\delta\cdots = \beta\delta\epsilon$. By the parity lemma, we have $\beta\cdots = \beta\delta\epsilon$. Therefore $\beta\gamma^2$ is not a vertex, a contradiction.

If $f = 12$, we get $\beta = \gamma = \frac{2}{3}$, a contradiction. If $f = 16$, by the equation (2.3), we have

$$\alpha = \tfrac{3}{4},\ \beta = 1,\ \gamma = \tfrac{1}{2},\ \tan(\delta\pi) = \tfrac{(5\sqrt{2}+4)\sqrt{886-572\sqrt{2}}}{68} - \tfrac{\sqrt{2}}{2} + \tfrac{1}{2} \approx 1.223.$$
$$\epsilon = 1 - \delta \approx 0.7182,\ \cos a = \tfrac{(2\sqrt{2}+5)\sqrt{443-286\sqrt{2}}}{68} + \tfrac{\sqrt{2}}{4} - \tfrac{1}{4} \approx 0.8182.$$

Here δ and ϵ are normalized by π, while a is in radians. Its $3D$ picture is shown in Figure 32. Therefore AVC $\subset \{\beta\delta\epsilon, \alpha^2\gamma, \alpha\beta\gamma, \beta\gamma^2, \delta^2\epsilon^2, \gamma^2\delta\epsilon\}$. The AAD of $\beta\gamma^2$ is $|^\alpha\beta^\delta|^\alpha\gamma^\epsilon|\gamma|$, $|^\alpha\beta^\delta|^\epsilon\gamma^\alpha|^\alpha\gamma^\epsilon|$ or $|^\alpha\beta^\delta|^\epsilon\gamma^\alpha|^\epsilon\gamma^\alpha|$. This implies $\alpha\delta\cdots$ or $\alpha\epsilon\cdots$ is a vertex, contradicting the AVC.

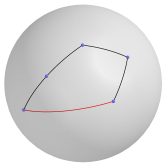


Figure 32: Case $\{\beta\delta\epsilon, \alpha^2\gamma, \beta\gamma^2\}$ for $f = 16$.

If $f \geq 20$, we get four solutions of δ to the equation (2.3), but none of them lies in $(0, 1)$. In other words, there is no such pentagon.

Case. $\{\beta\delta\epsilon, \alpha^2\gamma, \alpha\gamma^3\}$

The angle sums of $\beta\delta\epsilon$, $\alpha^2\gamma$, $\alpha\gamma^3$ and the angle sum for the pentagon imply

$$\alpha = \tfrac{4}{5},\ \beta + \delta + \epsilon = 2,\ \gamma = \tfrac{2}{5},\ f = 20.$$

Subcase. $\beta < \gamma$

By Lemma 10, we have $\delta > \epsilon$. By $\beta\delta\epsilon$, we can obtain that $\delta + \epsilon > \frac{8}{5}$ and $\delta > \frac{4}{5}$. By $R(\delta\epsilon\cdots)$, $R(\delta^2\cdots) <$ all angles and the parity lemma, we get $\delta\cdots = \beta\delta\epsilon$. Therefore AVC $\subset \{\beta\delta\epsilon, \alpha^2\gamma, \alpha\gamma^3, \gamma^5\}$.

The AAD is $\alpha^2\gamma = \alpha^\beta|^\beta\alpha\cdots$, $\alpha^\beta|^\gamma\alpha\cdots$ or $\alpha^\beta|^\alpha\gamma\cdots$. This implies $\alpha\beta\cdots$, $\beta^2\cdots$ or $\beta\gamma\cdots$ is a vertex, contradicting the AVC.

Subcase. $1 > \beta > \gamma$

By Lemma 10, we have $\delta < \epsilon$. By $\beta\delta\epsilon$ and $1 > \beta > \gamma$, we know $\delta + \epsilon > 1$. By $\delta < \epsilon$, we get $\epsilon > \frac{1}{2}$. By $R(\epsilon^2 \cdots) < \beta,\ 2\delta,\ \delta + \epsilon,\ 2\epsilon,\ 2\alpha,\ \alpha + \gamma,\ 3\gamma$, we get $\epsilon^2 \cdots = \gamma^2\epsilon^2$. If $\gamma^2\epsilon^2$ is a vertex, by the angle value, we have $\alpha = \frac{4}{5}$, $\beta + \delta = \frac{7}{5}$, $\gamma = \frac{2}{5}$, $\epsilon = \frac{3}{5}$. Applying the irrational angle lemma to this case, we have $n_2 = n_4$. Then $\beta \cdots = \beta\delta \cdots = \beta\delta\epsilon$. Therefore, $\gamma^2\epsilon^2$ is not a vertex. We get that $\epsilon^2 \cdots$ is not a vertex. By $R(\alpha\delta\epsilon \cdots) < \alpha,\ \beta,\ \gamma$ and the balance lemma, we get no $\alpha\epsilon \cdots$. The AAD of $\alpha\gamma^3$ is $|^\alpha\gamma^\epsilon|^\alpha\gamma^\epsilon|\gamma|$ or $|^\epsilon\gamma^\alpha|^\alpha\gamma^\epsilon|\gamma|$. This implies $\alpha\epsilon \cdots$ or $\epsilon^2 \cdots$ is a vertex, a contradiction.

Subcase. $\beta \geq 1 > \gamma$

By Lemma 10, we have $\delta < \epsilon$. By $\beta\delta\epsilon$ and $\beta \geq 1 > \gamma$, we know $\delta + \epsilon \leq 1$. By $\delta < \epsilon$, we get $\delta < \frac{1}{2}$. By the angle values, we get $\beta \cdots = \beta\delta\epsilon,\ \beta\delta^k,\ \alpha\beta\delta^l,\ \beta\gamma\delta^m$ or $\beta\gamma^2\delta^n$. If $\beta\gamma^2$ is a vertex, it actually belongs to Case. $\{\beta\delta\epsilon, \alpha^2\gamma, \beta\gamma^2\}$. By the balance lemma, we get $\beta \cdots = \beta\delta\epsilon$. Therefore, AVC $\subset \{\beta\delta\epsilon, \alpha^2\gamma, \alpha\gamma^3, \gamma^5\}$. We get the same contradiction as Subcase $\beta < \gamma$.

Similar to Case $\{\beta\delta\epsilon, \alpha^2\gamma, \alpha\gamma^3\}$, three other cases $\{\beta\delta\epsilon, \alpha^2\gamma, \alpha\gamma^2\}$, $\{\beta\delta\epsilon, \alpha^2\gamma,$ $\gamma^3\}$ and $\{\beta\delta\epsilon, \alpha^2\gamma, \alpha\gamma^4\}$ admit no tilings.

□

Proposition 6. *There is no non-symmetric a^4b-tiling with general angles, such that $\beta\delta\epsilon$ is the unique a^2b-vertex type and $\alpha\beta\gamma$ is a vertex.*

Proof. By Lemma 17, one of the vertices in the last column of Table 13: $\{\alpha^3,$ $\alpha^4,\ \alpha^3\beta,\ \alpha^3\gamma,\ \alpha^2\delta^2,\ \alpha^2\delta\epsilon,\ \alpha^2\epsilon^2,\ \alpha\gamma\delta^2,\ \delta^3\epsilon,\ \alpha^5,\ \alpha^4\beta,\ \alpha^4\gamma,\ \alpha^3\beta^2,\ \alpha^3\delta^2,\ \alpha^3\delta\epsilon,$ $\alpha\delta\epsilon^3,\ \alpha^3\epsilon^2,\ \alpha^2\beta\delta^2,\ \alpha^2\beta\epsilon^2,\ \alpha^2\gamma\delta^2,\ \alpha^2\gamma\epsilon^2,\ \alpha\gamma^2\delta^2,\ \alpha\delta^3\epsilon,\ \gamma\delta^4,\ \gamma\delta^3\epsilon\}$ must appear.

For each combination of 3 vertices, we will apply the irrational angle lemma and AAD to get Table 21. The proof is completed by the contradictions in the last column of Table 21 and the last row of Table 22.

□

Proposition 7. *There is no non-symmetric a^4b-tiling with general angles, such that $\beta\delta\epsilon$ is the unique a^2b-vertex type and $\alpha\gamma^2$ is a vertex.*

Proof. By Lemma 17, one of the vertices in the last column of Table 13: $\{\alpha^3,$ $\alpha^3\gamma,\ \alpha^2\beta\gamma,\ \alpha^2\delta\epsilon,\ \alpha\gamma\delta^2,\ \alpha\gamma\delta\epsilon,\ \alpha\gamma\epsilon^2,\ \delta^2\epsilon^2,\ \delta\epsilon^3,\ \alpha^4\gamma,\ \alpha^3\beta\gamma,\ \alpha^3\delta\epsilon,\ \alpha^2\beta^2\gamma,\ \alpha^2\gamma\delta^2,$ $\alpha^2\gamma\delta\epsilon,\ \alpha^2\gamma\epsilon^2,\ \alpha\beta\gamma\epsilon^2,\ \alpha\delta^3\epsilon,\ \alpha\delta^2\epsilon^2,\ \alpha\delta\epsilon^3,\ \gamma\delta^3\epsilon,\ \gamma\delta^2\epsilon^2,\ \gamma\delta\epsilon^3\}$ must appear.

For each combination of 3 vertices, we will apply the irrational angle lemma and AAD to get Table 23. The proof is completed by Table 23, 24, 25, 26 and 27, similar to Proposition 1.

□

Proposition 8. *There is no non-symmetric a^4b-tiling with general angles, such that $\beta\delta\epsilon$ is the unique a^2b-vertex type and α^3 is a vertex.*

Table 21: $\beta\delta\epsilon$, $\alpha\beta\gamma$ are vertices.

Vertex		Angles	Irrational Angle Lemma	AAD	Contradiction
$\beta\delta\epsilon, \alpha\beta\gamma$	$\alpha^2\epsilon^2$	$(\delta-\frac{4}{f}, 1-\frac{4}{f}, 1-\delta+\frac{8}{f}, \delta, 1-\delta+\frac{4}{f})$	$n_1+n_4=n_3+n_5$	$\lvert\gamma\epsilon^\delta\rvert^\delta\epsilon\gamma\rvert\cdots$	No $\delta^2\cdots$ by the irrational angle lemma
	$\alpha^2\beta\epsilon^2$	$(-\frac{1}{2}+\delta-\frac{2}{f}, 1-\frac{4}{f}, \frac{3}{2}-\delta+\frac{6}{f}, \delta, 1-\delta+\frac{4}{f})$			
	$\alpha^3\epsilon^2$	$(\frac{2}{3}\delta-\frac{8}{3f}, 1-\frac{4}{f}, 1-\frac{2}{3}\delta+\frac{20}{3f}, \delta, 1-\delta+\frac{4}{f})$	$2n_1+3n_4=2n_3+3n_5$		
	$\alpha^2\gamma\epsilon^2$	$(-1+2\delta-\frac{12}{f}, 1-\frac{4}{f}, 2-2\delta+\frac{16}{f}, \delta, 1-\delta+\frac{4}{f})$	$2n_1+n_4=2n_3+n_5$		
	$\alpha\delta\epsilon^3$				
	$\alpha^2\delta^2$	$(1-\delta, 1-\frac{4}{f}, \delta+\frac{4}{f}, \delta, 1-\delta+\frac{4}{f})$	$n_1+n_5=n_3+n_4$	$\lvert\beta\delta^\epsilon\rvert^\epsilon\delta^\beta\rvert\cdots$	No $\epsilon^2\cdots$
	$\alpha^2\beta\delta^2$	$(\frac{1}{2}-\delta+\frac{2}{f}, 1-\frac{4}{f}, \frac{1}{2}+\delta+\frac{2}{f}, \delta, 1-\delta+\frac{4}{f})$			
	$\alpha^3\delta^2$	$(\frac{2}{3}-\frac{2}{3}\delta, 1-\frac{4}{f}, \frac{1}{3}+\frac{2}{3}\delta+\frac{4}{f}, \delta, 1-\delta+\frac{4}{f})$	$2n_1+3n_5=2n_3+3n_4$		
	$\alpha\gamma\delta^2$	$(1-\gamma+\frac{4}{f}, 1-\frac{4}{f}, \gamma, \frac{1}{2}-\frac{2}{f}, \frac{1}{2}+\frac{6}{f})$	$n_1=n_3$		
	$\delta^3\epsilon$				
	$\alpha^2\gamma\delta^2$	$(1-2\delta-\frac{4}{f}, 1-\frac{4}{f}, 2\delta+\frac{8}{f}, \delta, 1-\delta+\frac{4}{f})$	$2n_1+n_5=2n_3+n_4$		
	$\alpha\delta^3\epsilon$				
	$\alpha\gamma^2\delta^2$	$(2\delta+\frac{8}{f}, 1-\frac{4}{f}, 1-2\delta-\frac{4}{f}, \delta, 1-\delta+\frac{4}{f})$	$2n_1+n_4=2n_3+n_5$		
	$\gamma\delta^3\epsilon$				
	$\gamma\delta^4$	$(-1+4\delta+\frac{4}{f}, 1-\frac{4}{f}, 2-4\delta, \delta, 1-\delta+\frac{4}{f})$	$4n_1+n_4=4n_3+n_5$		
	α^4	$(\frac{1}{2}, 1-\frac{4}{f}, \frac{1}{2}+\frac{4}{f}, \delta, 1-\delta+\frac{4}{f})$	$n_4=n_5$		No degree ≥ 4 vertex $\delta\epsilon\cdots$ (similar to Proposition 4)
	$\alpha^3\beta$	$(\frac{1}{3}+\frac{4}{3f}, 1-\frac{4}{f}, \frac{2}{3}+\frac{8}{3f}, \delta, 1-\delta+\frac{4}{f})$			
	α^5	$(\frac{2}{5}, 1-\frac{4}{f}, \frac{3}{5}+\frac{4}{f}, \delta, 1-\delta+\frac{4}{f})$			
	$\alpha^4\beta$	$(\frac{1}{4}+\frac{1}{f}, 1-\frac{4}{f}, \frac{3}{4}+\frac{3}{f}, \delta, 1-\delta+\frac{4}{f})$			
	α^3	$(\frac{2}{3}, 1-\frac{4}{f}, \frac{1}{3}+\frac{4}{f}, \delta, 1-\delta+\frac{4}{f})$			See Table 22
	$\alpha^3\gamma$	$(\frac{1}{2}-\frac{2}{f}, 1-\frac{4}{f}, \frac{1}{2}+\frac{6}{f}, \delta, 1-\delta+\frac{4}{f})$			
	$\alpha^2\delta\epsilon$				
	$\alpha^4\gamma$	$(\frac{1}{3}-\frac{4}{3f}, 1-\frac{4}{f}, \frac{2}{3}+\frac{16}{3f}, \delta, 1-\delta+\frac{4}{f})$			
	$\alpha^3\delta\epsilon$				
	$\alpha^3\beta^2$	$(\frac{8}{3f}, 1-\frac{4}{f}, 1+\frac{4}{3f}, \delta, 1-\delta+\frac{4}{f})$			

Table 22: AVC for Cases $\{\beta\delta\epsilon, \alpha\beta\gamma, \alpha^3\}$, $\{\beta\delta\epsilon, \alpha\beta\gamma, \alpha^3\gamma(\alpha^2\delta\epsilon)\}$, $\{\beta\delta\epsilon, \alpha\beta\gamma, \alpha^4\gamma(\alpha^3\delta\epsilon)\}$ and $\{\beta\delta\epsilon, \alpha\beta\gamma, \alpha^3\beta^2\}$.

AVC for $\{\beta\delta\epsilon, \alpha\beta\gamma, \alpha^3\}$		AVC for $\{\beta\delta\epsilon, \alpha\beta\gamma, \alpha^3\gamma(\alpha^2\delta\epsilon)\}$		AVC for $\{\beta\delta\epsilon, \alpha\beta\gamma, \alpha^4\gamma(\alpha^3\delta\epsilon)\}$		AVC for $\{\beta\delta\epsilon, \alpha\beta\gamma, \alpha^3\beta^2\}$	
f	vertex	f	vertex	f	vertex	f	vertex
all	$\beta\delta\epsilon, \alpha\beta\gamma, \alpha^3$	all	$\beta\delta\epsilon, \alpha\beta\gamma, \alpha^3\gamma, \alpha^2\delta\epsilon$	all	$\beta\delta\epsilon, \alpha\beta\gamma, \alpha^4\gamma, \alpha^3\delta\epsilon$	all	$\beta\delta\epsilon, \alpha\beta\gamma, \alpha^3\beta^2$
36	$\gamma^2\delta\epsilon, \alpha\gamma^3$	36	γ^3	16	$\alpha^2\beta^2, \alpha^5\beta, \alpha^8$	$8s, s=2,3,\cdots$	α^{6s}
						$8s+4, s=2,3,\cdots$	$\alpha^{6s+3}, \alpha^{3s+3}\beta, \alpha^{3s+1}\gamma, \alpha^{3s}\delta\epsilon$
By the balance lemma, $\gamma^2\delta\epsilon$ must appear. Its AAD gives $\alpha\lvert\epsilon$, $\beta\lvert\beta$ or $\epsilon\lvert\epsilon$, contradicting the AVC		By the balance lemma, $\alpha^2\delta\epsilon$ must appear. Its AAD gives $\beta\lvert\beta$, $\gamma\lvert\delta$ or $\gamma\lvert\epsilon$, contradicting the AVC		Can not balance.			

Table 23: $\beta\delta\epsilon, \alpha\gamma^2$ are vertices.

Vertex		Angles	Irrational Angle Lemma	AAD	Contradiction
$\beta\delta\epsilon, \alpha\gamma^2$	$\alpha\gamma\epsilon^2$	$(\frac{8}{f}, \frac{3}{2}-\delta+\frac{2}{f}, 1-\frac{4}{f}, \delta, \frac{1}{2}-\frac{2}{f})$	$n_2=n_4$	$\lvert\gamma\epsilon^\delta\rvert^\delta\epsilon\gamma\rvert\cdots$	No $\delta^2\cdots$ by the irrational angle lemma
	$\alpha^2\gamma\epsilon^2$	$(\frac{8}{f}, \frac{3}{2}-\delta+\frac{6}{f}, 1-\frac{4}{f}, \delta, \frac{1}{2}-\frac{6}{f})$			
	$\delta\epsilon^3$	$(\frac{8}{f}, \frac{3}{2}-\frac{2}{3}\delta, 1-\frac{4}{f}, \delta, \frac{2}{3}-\frac{1}{3}\delta)$	$2n_2+n_5=3n_4$		
	$\alpha\delta\epsilon^3$	$(\frac{8}{f}, \frac{4}{3}-\frac{2}{3}\delta+\frac{8}{3f}, 1-\frac{4}{f}, \delta, \frac{2}{3}-\frac{1}{3}\delta-\frac{8}{3f})$			
	$\gamma\delta\epsilon^3$	$(\frac{8}{f}, \frac{5}{3}-\frac{2}{3}\delta-\frac{4}{3f}, 1-\frac{4}{f}, \delta, \frac{3}{2}-\frac{1}{3}\delta+\frac{4}{3f})$			
	$\alpha\beta\gamma\epsilon^2$	$(\frac{8}{f}, 3-2\delta+\frac{4}{f}, 1-\frac{4}{f}, \delta, -1+\delta-\frac{2}{f})$	$2n_2=n_4+n_5$		
	$\alpha\gamma\delta^2$	$(\frac{8}{f}, \frac{3}{2}-\epsilon+\frac{2}{f}, 1-\frac{4}{f}, \frac{1}{2}-\frac{2}{f}, \epsilon)$	$n_2=n_5$	$\lvert\beta\delta^\epsilon\rvert^\epsilon\delta^\beta\rvert\cdots$	No $\epsilon^2\cdots$
	$\alpha^2\gamma\delta^2$	$(\frac{8}{f}, \frac{3}{2}-\epsilon+\frac{6}{f}, 1-\frac{4}{f}, \frac{1}{2}-\frac{6}{f}, \epsilon)$			
	$\alpha\delta^3\epsilon$	$(\frac{8}{f}, \frac{4}{3}-\frac{2}{3}\epsilon+\frac{8}{3f}, 1-\frac{4}{f}, \frac{2}{3}-\frac{1}{3}\epsilon-\frac{8}{3f}, \epsilon)$	$2n_2+n_4=3n_5$		
	$\gamma\delta^3\epsilon$	$(\frac{8}{f}, 1+2\delta-\frac{4}{f}, 1-\frac{4}{f}, \delta, 1-3\delta+\frac{4}{f})$			
	$\alpha\gamma\delta\epsilon$	$(\frac{8}{f}, 1+\frac{4}{f}, 1-\frac{4}{f}, \delta, 1-\delta-\frac{4}{f})$	$n_4=n_5$		See Table 24
	$\alpha\delta^2\epsilon^2$				
	$\delta^2\epsilon^2$	$(\frac{8}{f}, 1, 1-\frac{4}{f}, \delta, 1-\delta)$			
	$\alpha^2\beta\gamma$	$(\frac{8}{f}, 1-\frac{12}{f}, 1-\frac{4}{f}, \delta, 1-\delta+\frac{12}{f})$			See Table 25
	$\alpha^3\beta\gamma$	$(\frac{8}{f}, 1-\frac{20}{f}, 1-\frac{4}{f}, \delta, 1-\delta+\frac{20}{f})$			
	$\alpha^2\beta^2\gamma$	$(\frac{8}{f}, \frac{1}{2}-\frac{6}{f}, 1-\frac{4}{f}, \delta, \frac{3}{2}-\delta+\frac{6}{f})$			
	$\alpha^2\delta\epsilon$	$(\frac{8}{f}, \frac{16}{f}, 1-\frac{4}{f}, \delta, 2-\delta-\frac{16}{f})$			See Table 26
	$\alpha^3\delta\epsilon$	$(\frac{8}{f}, \frac{24}{f}, 1-\frac{4}{f}, \delta, 2-\delta-\frac{24}{f})$			
	$\alpha^2\gamma\delta\epsilon$	$(\frac{8}{f}, 1+\frac{12}{f}, 1-\frac{4}{f}, \delta, 1-\delta-\frac{12}{f})$			See Table 27
	$\gamma\delta^2\epsilon^2$	$(\frac{8}{f}, \frac{3}{2}-\frac{2}{f}, 1-\frac{4}{f}, \delta, \frac{1}{2}-\delta+\frac{2}{f})$			
	α^3	$(\frac{2}{3}, 2-\delta-\epsilon, \frac{2}{3}, \delta, \epsilon)$, $f=12$			Similar to Proposition 5 Case $\{\beta\delta\epsilon, \alpha^2\gamma, \alpha\gamma^3\}$
	$\alpha^3\gamma$	$(\frac{2}{5}, 2-\delta-\epsilon, \frac{3}{5}, \delta, \epsilon)$, $f=20$			
	$\alpha^4\gamma$	$(\frac{2}{7}, 2-\delta-\epsilon, \frac{5}{7}, \delta, \epsilon)$, $f=28$			

Table 24: AVC for Cases $\{\beta\delta\epsilon, \alpha\gamma^2, \alpha\gamma\delta\epsilon(\alpha\delta^2\epsilon^2)\}$, $\{\beta\delta\epsilon, \alpha\gamma^2, \delta^2\epsilon^2\}$ and $\{\beta\delta\epsilon, \alpha\gamma^2, \alpha^2\beta\gamma\}$.

AVC for $\{\beta\delta\epsilon, \alpha\gamma^2, \alpha\gamma\delta\epsilon(\alpha\delta^2\epsilon^2)\}$		AVC for $\{\beta\delta\epsilon, \alpha\gamma^2, \delta^2\epsilon^2\}$		AVC for $\{\beta\delta\epsilon, \alpha\gamma^2, \alpha^2\beta\gamma\}$	
f	vertex	f	vertex	f	vertex
all	$\beta\delta\epsilon, \alpha\gamma^2, \alpha\gamma\delta\epsilon, \alpha\delta^2\epsilon^2$	all	$\beta\delta\epsilon, \alpha\gamma^2, \delta^2\epsilon^2$	all	$\beta\delta\epsilon, \alpha\gamma^2, \alpha^2\beta\gamma, \alpha^3\beta^2$
$8s+4, s=2,3,\cdots$	$\alpha^{2s+1}, \alpha^s\beta, \alpha^{s+1}\gamma, \alpha^{s+1}\delta\epsilon$	$8s, s=2,3,\cdots$	$\alpha^{2s}, \alpha^s\beta, \alpha^s\delta\epsilon$	28	$\beta^2\gamma, \alpha^2\delta\epsilon, \alpha\beta^3, \alpha^4\gamma, \alpha^5\beta, \alpha^7$
				36	$\beta^3, \alpha^3\delta\epsilon, \alpha^5\gamma, \alpha^6\beta, \alpha^9$
				$8s+4, s=5,6,\cdots$	$\alpha^{2s+1}, \alpha^{s+2}\beta, \alpha^{s+1}\gamma, \alpha^{s-1}\delta\epsilon$
By the balance lemma, $\alpha^s\beta$ must appear. Its AAD gives $\beta\|\beta$ or $\beta\|\gamma$, contradicting the AVC				By the balance lemma, $\alpha^k\delta\epsilon$ must appear. Its AAD gives $\gamma\|\delta$ or $\gamma\|\epsilon$, contradicting the AVC	

Table 25: AVC for Cases $\{\beta\delta\epsilon, \alpha\gamma^2, \alpha^3\beta\gamma\}$ and $\{\beta\delta\epsilon, \alpha\gamma^2, \alpha^2\beta^2\gamma\}$.

AVC for $\{\beta\delta\epsilon, \alpha\gamma^2, \alpha^3\beta\gamma\}$		AVC for $\{\beta\delta\epsilon, \alpha\gamma^2, \alpha^2\beta^2\gamma\}$	
f	vertex	f	vertex
all	$\beta\delta\epsilon, \alpha\gamma^2, \alpha^3\beta\gamma, \alpha^5\beta^2$	all	$\beta\delta\epsilon, \alpha\gamma^2, \alpha^2\beta^2\gamma, \alpha^3\beta^4$
36	$\alpha^2\delta\epsilon, \alpha\beta^2\gamma, \alpha\beta^4, \alpha^5\gamma, \alpha^3\beta^3, \alpha^7\beta, \alpha^9$	44	$\alpha^2\delta\epsilon, \beta^3\gamma, \alpha^4\beta\gamma, \alpha\beta^5, \alpha^6\gamma, \alpha^5\beta^3, \alpha^7\beta^2, \alpha^9\beta, \alpha^{11}$
44	$\beta^2\gamma, \alpha^3\delta\epsilon, \alpha^2\beta^3, \alpha^6\gamma, \alpha^8\beta, \alpha^{11}$	60	$\alpha^3\delta\epsilon, \beta^5, \alpha^5\beta\gamma, \alpha^8\gamma, \alpha^6\beta^3, \alpha^9\beta^2, \alpha^{12}\beta, \alpha^{15}$
52	$\alpha\beta^3, \alpha^4\delta\epsilon, \alpha^7\gamma, \alpha^9\beta, \alpha^{13}$,	$16s+12, s=4,5,\cdots$	$\alpha^{4s+3}, \alpha^{3s+3}\beta, \alpha^{2s+3}\beta^2, \alpha^{s+3}\beta^3, \alpha^{s+2}\beta\gamma$,
60	$\beta^3, \alpha^5\delta\epsilon, \alpha^8\gamma, \alpha^{10}\beta, \alpha^{15}$		$\alpha^{2s+2}\gamma, \alpha^s\delta\epsilon$
$8s+4, s=8,9,\cdots$	$\alpha^{2s+1}, \alpha^{s+3}\beta, \alpha^{s+1}\gamma, \alpha^{s-2}\delta\epsilon$		
By the balance lemma, $\alpha^k\delta\epsilon$ must appear. Its AAD gives $\gamma\|\delta$ or $\gamma\|\epsilon$, contradicting the AVC			

Table 26: AVC for Cases $\{\beta\delta\epsilon, \alpha\gamma^2, \alpha^2\delta\epsilon\}$ and $\{\beta\delta\epsilon, \alpha\gamma^2, \alpha^3\delta\epsilon\}$.

AVC for $\{\beta\delta\epsilon, \alpha\gamma^2, \alpha^2\delta\epsilon\}$		AVC for $\{\beta\delta\epsilon, \alpha\gamma^2, \alpha^3\delta\epsilon\}$	
f	vertex	f	vertex
all	$\beta\delta\epsilon, \alpha\gamma^2, \alpha^2\delta\epsilon$	all	$\beta\delta\epsilon, \alpha\gamma^2, \alpha^3\delta\epsilon$
16	$\alpha^2\beta, \alpha^4, \delta^2\epsilon^2$	20	$\alpha^2\beta, \alpha^3\gamma, \alpha\gamma\delta\epsilon, \alpha^5, \alpha\delta^2\epsilon^2$
$8s, s=3,4,\cdots$	$\alpha^{2s-2x}\beta^x (x=0,1,\cdots,s)$	24	$\alpha^3\beta, \delta^2\epsilon^2, \alpha^6$
$8s+4, s=3,4,\cdots$	$\alpha^{2s+1-2x}\beta^x (x=0,1,\cdots,s)$,	$8s, s=4,5,\cdots$	$\alpha^{2s-3x}\beta^x (x=0,1,\cdots,\lfloor\frac{2s}{3}\rfloor))$
	$\alpha^{s+1-2y}\beta^y\gamma (y=0,1,\cdots,\lfloor\frac{s-1}{2}\rfloor)$	$8s+4, s=3,4,\cdots$	$\alpha^{2s+1-3x}\beta^x (x=0,1,\cdots,\lfloor\frac{2s+1}{3}\rfloor))$,
			$\alpha^{s+1-3y}\beta^y\gamma (y=0,1,\cdots,\lfloor\frac{s-1}{3}\rfloor)$
By the balance lemma, $\alpha^k\beta^l$ or $\alpha^k\beta^l\gamma$ must appear. Its AAD gives $\beta\|\beta$, $\beta\|\gamma$, $\gamma\|\delta$, $\gamma\|\epsilon$, $\delta\|\delta$, $\delta\|\epsilon$ or $\epsilon\|\epsilon$, contradicting the AVC			

Table 27: AVC for Cases $\{\beta\delta\epsilon, \alpha\gamma^2, \alpha^2\gamma\delta\epsilon\}$ and $\{\beta\delta\epsilon, \alpha\gamma^2, \gamma\delta^2\epsilon^2\}$.

AVC for $\{\beta\delta\epsilon, \alpha\gamma^2, \alpha^2\gamma\delta\epsilon\}$		AVC for $\{\beta\delta\epsilon, \alpha\gamma^2, \gamma\delta^2\epsilon^2\}$	
f	vertex	f	vertex
all	$\beta\delta\epsilon, \alpha\gamma^2, \alpha^2\gamma\delta\epsilon, \alpha^3\delta^2\epsilon^2$	all	$\beta\delta\epsilon, \alpha\gamma^2, \gamma\delta^2\epsilon^2$
28	$\alpha^2\beta, \alpha^4\gamma, \gamma\delta^2\epsilon^2, \alpha^7, \alpha^5\delta\epsilon, \alpha\delta^3\epsilon^3$	$16s+12, s=1,2,\cdots$	$\alpha^{4s+3}, \alpha^{s+1}\beta, \alpha^{2s+2}\gamma, \alpha^{s+1}\gamma\delta\epsilon$,
36	$\alpha^3\beta, \alpha^5\gamma, \delta^3\epsilon^3, \alpha^6\delta\epsilon, \alpha^9$		$\alpha^{3s+2}\delta\epsilon, \alpha^{2s+1}\delta^2\epsilon^2, \alpha^s\delta^3\epsilon^3$
$8s+4, s=5,6,\cdots$	$\alpha^{2s+1}, \alpha^{s-1}\beta, \alpha^{s+1}\gamma, \alpha^{s+2}\delta\epsilon$		
By the balance lemma, $\alpha^k\beta$ must appear. Its AAD gives $\beta\|\beta$ or $\beta\|\gamma$, contradicting the AVC			

Proof. By Lemma 17, one of the vertices in the last column of Table 13: $\{\alpha\beta^2, \alpha\beta\gamma, \alpha\gamma^2\}$ must appear. By Proposition 6, 7, we may assume that $\alpha\beta\gamma$, $\alpha\gamma^2$ are not vertices. So $\alpha\beta^2$ is a vertex. By the irrational angle lemma, we have $n_4 = n_5$. The angle sums of $\beta\delta\epsilon, \alpha^3, \alpha\beta^2$ and the angle sum for the pentagon imply

$$\alpha = \tfrac{2}{3},\ \beta = \tfrac{2}{3},\ \gamma = \tfrac{1}{3} + \tfrac{4}{f},\ \delta + \epsilon = \tfrac{4}{3}.$$

By $\beta \neq \gamma$, we get $f \geq 14$. By $\gamma < R(\delta\epsilon\cdots) = \alpha = \beta < 2\gamma, \delta + \epsilon$ and $n_4 = n_5$, we have $\delta\cdots = \delta\epsilon\cdots = \beta\delta\epsilon$ since $\beta\delta\epsilon$ is the unique a^2b-vertex type. Then $\#\beta \geq \#\beta\delta\epsilon + \#\alpha\beta^2 = f + \#\alpha\beta^2 > f$, a contradiction. □

6 Tilings with a unique a^2b-vertex type $\alpha\delta\epsilon$

This section is devoted to the classification of edge-to-edge tilings of the sphere by congruent pentagons with the edge combination a^4b, where $\alpha\delta\epsilon$ is the unique a^2b-vertex type. By Lemma 7, we know that there exists an a^3-vertex around the special tile. This means one of $\{\alpha^3,\ \beta^3,\ \gamma^3,\ \alpha^2\beta,\ \alpha^2\gamma,\ \alpha\beta^2,\ \alpha\gamma^2,\ \beta^2\gamma,\ \beta\gamma^2,\ \alpha\beta\gamma\}$ must appear. Up to the symmetry of interchanging $\beta \leftrightarrow \gamma$ and $\delta \leftrightarrow \epsilon$, it is enough to consider the following cases $\{\alpha^3, \alpha^2\gamma, \alpha\gamma^2, \alpha\beta\gamma, \beta^2\gamma, \gamma^3\}$.

Proposition 9. *In a non-symmetric a^4b-tiling with general angles, if $\alpha\delta\epsilon$ is the unique a^2b-vertex type and $\beta^2\gamma$ is a vertex, then the tiling is the non-symmetric 3-layer earth map tiling $T(4m\,\alpha\delta\epsilon, 2m\,\beta^2\gamma, 2\gamma^m)$ with $4m$ tiles for some $m \geq 4$, or its standard flip modification $T((8k+4)\alpha\delta\epsilon, 4k\,\beta^2\gamma, 4\beta\gamma^{k+1})$ for odd $m = 2k+1$.*

Proof. The angle sums of $\alpha\delta\epsilon$, $\beta^2\gamma$ and the angle sum for the pentagon imply

$$\alpha + \delta + \epsilon = 2,\ \beta = 1 - \tfrac{4}{f},\ \gamma = \tfrac{8}{f}.$$

We have $1 > \beta > \frac{2}{3} > \gamma$. By Lemma 10, this implies $\delta < \epsilon$. We divide the discussion into three cases.

Case. $\alpha \geq 1$

Then $\delta, \epsilon < 1$. By Lemma 9, we get $\beta + 2\delta > 1$ and $\gamma + 2\epsilon > 1$. By $R(\alpha\cdots) = \delta + \epsilon < \beta + \gamma, \beta + 2\delta, \gamma + 2\epsilon$ and the parity lemma, we get $\alpha\cdots = \alpha\delta\epsilon, \alpha\gamma^x (x \geq 2)$ or $\alpha\gamma^y\delta^z (z \geq 2)$.

If $\alpha\gamma^x$ is a vertex, by the irrational angle lemma and $\alpha\delta\epsilon$, we have $n_4 = n_5$ and $\alpha\gamma^y\delta^z$ cannot be a vertex. Then we have the unique AAD $\alpha\gamma\cdots = \alpha\gamma^x = |^\gamma\alpha^\beta|^\epsilon\gamma^\alpha|\ldots|^\epsilon\gamma^\alpha|$ since $\alpha^2\cdots$ and $\alpha\beta\cdots$ are not vertices. This gives a vertex $\beta^\alpha|^\gamma\epsilon\cdots$. By $\beta + 2\delta > 1$ and $\gamma + 2\epsilon > 1$, we get $R(\beta\delta\epsilon\cdots) < \gamma + \delta + \epsilon$. By $R(\beta^2\cdots) = \gamma < x\gamma = \delta + \epsilon$, we get $\beta^2\cdots = \beta^2\gamma$. Then $\beta\epsilon\cdots = \beta\delta\epsilon\cdots = \beta\gamma^y\delta\epsilon$ or $\beta\delta^2\epsilon^2$. Both possibilities will lead to some contradiction in the following.

By $\alpha|\gamma\cdots = \alpha^\beta|^\epsilon\gamma\cdots$ or $\alpha^\gamma|^\alpha\gamma\cdots$, we have the AAD $\beta|\epsilon\cdots = \beta\delta^2\epsilon^2 = |^\delta\beta^\alpha|^\gamma\epsilon^\delta|^\epsilon\delta^\beta|^\beta\delta^\epsilon|^\delta\epsilon^\gamma|$, $|^\delta\beta^\alpha|^\gamma\epsilon^\delta|^\epsilon\delta^\beta|^\gamma\epsilon^\delta|^\epsilon\delta^\beta|$ or $|^\delta\beta^\alpha|^\gamma\epsilon^\delta|^\delta\epsilon^\gamma|^\beta\delta^\epsilon|^\epsilon\delta^\beta|$. This gives a vertex $\beta^\delta|^\delta\beta\cdots$ or $|^\alpha\beta^\delta|^\epsilon\gamma^\alpha|\cdots$. Then $\beta^\delta|^\delta\beta\cdots = \beta^2\gamma = |^\alpha\beta^\delta|^\delta\beta^\alpha|^\alpha\gamma^\epsilon|$. This

gives a vertex $\alpha^2\cdots$, a contradiction. Then $|^\alpha\beta^\delta|^\epsilon\gamma^\alpha|\cdots=\beta^2\gamma$, $\beta\gamma^{\frac{f+4}{8}}$ or $\beta\gamma^y\delta\epsilon$. Either one gives a vertex $\alpha^2\cdots$ or $\alpha\beta\cdots$, a contradiction. Therefore, $\beta\delta^2\epsilon^2$ is not a vertex.

We have the unique AAD $\beta|\epsilon\cdots=\beta\gamma^y\delta\epsilon=|^\delta\beta^\alpha|^\gamma\epsilon^\delta\rceil^\epsilon\delta^\beta|^\epsilon\gamma^\alpha|\cdots|^\epsilon\gamma^\alpha|$. This gives a new $|^\alpha\beta^\delta|^\gamma\epsilon^\delta\rceil\cdots=\beta\gamma^y\delta\epsilon=|^\alpha\beta^\delta|^\gamma\epsilon^\delta\rceil^\epsilon\delta^\beta|\gamma|\cdots$. This gives a vertex $\alpha^2\cdots$ or $\alpha\beta\cdots$, a contradiction. Therefore, $\beta\gamma^y\delta\epsilon$ is not a vertex.

In summary, $\alpha\gamma^x$ is not a vertex and $\alpha\cdots=\alpha\delta\epsilon$ or $\alpha\gamma^y\delta^z(z\geq 2)$. To balance α and δ, we have $\alpha\cdots=\alpha\delta\epsilon$. By the angle values, we have AVC $\subset\{\alpha\delta\epsilon,\beta^2\gamma,\beta\gamma^{\frac{f+4}{8}},\gamma^{\frac{f}{4}}\}$.

Case. $\alpha<1$ and $\epsilon\geq 1$

By $R(\epsilon\cdots)=\alpha+\delta<1<2\beta,\beta+\gamma$, the parity lemma and the balance lemma, we get $\epsilon\cdots=\delta\cdots=\alpha\delta\epsilon$ or $\gamma^x\delta\epsilon(x\geq 2)$. If $\gamma^x\delta\epsilon$ is a vertex, we have the unique AAD $\gamma^x\delta\epsilon=|^\gamma\epsilon^\delta\rceil^\epsilon\delta^\beta|^\alpha\gamma^\epsilon|\cdots|^\alpha\gamma^\epsilon|$ since $\beta\epsilon\cdots$ and $\epsilon^2\cdots$ are not vertices. This gives a vertex $|^\beta\alpha^\gamma|^\delta\beta^\alpha|\cdots$. By $\alpha=x\gamma\geq 2\gamma$ and $\gamma|\epsilon\cdots=\gamma^x\delta\epsilon=\gamma^\epsilon|^\gamma\epsilon\cdots$, we have the AAD $|^\beta\alpha^\gamma|^\delta\beta^\alpha|\cdots=\alpha^y\beta\gamma^z(z\geq 0)=|\alpha|\cdots|^\beta\alpha^\gamma|^\delta\beta^\alpha|^\epsilon\gamma^\alpha|\cdots|^\epsilon\gamma^\alpha|$.

By $R(\beta^2\cdots)=\gamma$ and $\alpha=x\gamma\geq 2\gamma$, we get $\beta^2\cdots=\beta^2\gamma$. Then we get the unique AAD $\beta^2\gamma=|^\alpha\beta^\delta|^\alpha\beta^\delta|^\alpha\gamma^\epsilon|$ since $\delta^2\cdots$ and $\delta|\epsilon\cdots$ are not vertices. Therefore there is no $\alpha^\beta|^\beta\alpha\cdots$ and $\beta\gamma\cdots=\beta^2\gamma$ or $\alpha^y\beta\gamma^z(y\geq 0)$.

Then AAD $\alpha^\beta|^\gamma\alpha\cdots$ implies a vertex $|^\delta\beta^\alpha|^\alpha\gamma^\epsilon|\cdots=\alpha^y\beta\gamma^z$. This gives a vertex $\beta\epsilon\cdots$ or $\gamma^\epsilon|^\gamma\alpha\cdots$, or $\epsilon|\epsilon\cdots$, or $\delta|\epsilon\cdots$, a contradiction. Therefore there is no $\alpha^\beta|^\gamma\alpha\cdots$. This implies that there is no $\alpha|\alpha|\alpha\cdots$ and $\alpha|\alpha\cdots=\alpha^\gamma|^\gamma\alpha\cdots$.

Then we have the unique AAD $|^\beta\alpha^\gamma|^\delta\beta^\alpha|\cdots=\alpha\beta\gamma^z=|^\beta\alpha^\gamma|^\delta\beta^\alpha|^\epsilon\gamma^\alpha|\cdots|^\epsilon\gamma^\alpha|$. Similarly, we have the unique AAD $\alpha|\beta\cdots=\alpha^y\beta\gamma^z=\alpha|^\delta\beta\cdots=|^\beta\alpha^\gamma|^\delta\beta^\alpha|^\epsilon\gamma^\alpha|\cdots|^\epsilon\gamma^\alpha|$. However, the AAD $|^\beta\alpha^\gamma|^\delta\beta^\alpha|^\epsilon\gamma^\alpha|\cdots|^\epsilon\gamma^\alpha|$ implies a new $|^\beta\alpha^\gamma|^\alpha\beta^\delta|\cdots$. The contradiction implies $\alpha^\gamma|^\delta\beta\cdots$ is not a vertex.

In summary, $\gamma^x\delta\epsilon$ is not a vertex. Then $\epsilon\cdots=\alpha\delta\epsilon$. By the angle values, we have AVC $\subset\{\alpha\delta\epsilon,\beta^2\gamma,\beta\gamma^{\frac{f+4}{8}},\gamma^{\frac{f}{4}}\}$.

Case. $\alpha<1$ and $\epsilon<1$

Then it is convex and $\alpha+\delta>1$, $\delta+\epsilon>1$. By Lemma 9, we get $\alpha+2\gamma>1$, $2\alpha+\gamma>1$, $2\alpha+\beta>1$ and $\beta+2\delta>1$. By $R(\epsilon^2\cdots)<\alpha,\beta+\gamma,2\delta$ and the parity lemma, we get $\epsilon^2\cdots=\gamma^x\epsilon^2(x\geq 2)$. If $\gamma^x\epsilon^2$ is a vertex, by the irrational angle lemma, we have $n_1=n_4$. Then AAD $\gamma^x\epsilon^2=\epsilon^\delta\rceil^\delta\epsilon\cdots$ implies that $\delta^2\cdots$ is a vertex. By $n_1=n_4$, we have $\delta^2\cdots=\alpha^2\delta^2\cdots$, contradicting $2\alpha+2\delta>2$. Therefore $\gamma^x\epsilon^2$ is not a vertex. By $R(\delta\epsilon\cdots)=\alpha<1<2\beta,\beta+\gamma$ and the parity lemma, we get $\epsilon\cdots=\delta\epsilon\cdots=\alpha\delta\epsilon$ or $\gamma^y\delta\epsilon(y\geq 2)$. Then the same proof as Case $\alpha<1$ and $\epsilon\geq 1$ gives AVC $\subset\{\alpha\delta\epsilon,\beta^2\gamma,\beta\gamma^{\frac{f+4}{8}},\gamma^{\frac{f}{4}}\}$.

In all three cases above, we get AVC $\subset\{\alpha\delta\epsilon,\beta^2\gamma,\beta\gamma^{\frac{f+4}{8}},\gamma^{\frac{f}{4}}\}$. Then we get the non-symmetric 3-layer earth map tiling and its standard flip modification by exactly the same proof as the "Subcase. $\gamma^{\frac{f}{4}}$ is a vertex" and "Subcase. $\beta\gamma^{\frac{f+4}{8}}$ is a vertex" on page 18.

□

By Proposition 9, we can assume that $\beta^2\gamma$ and $\beta\gamma^2$ are not vertices anymore in the rest of this section.

Proposition 10. *In a non-symmetric a^4b-tiling with general angles, if $\alpha\delta\epsilon$ is the unique a^2b-vertex type and γ^3 is a vertex, then all tilings are*

1. *Pentagonal subdivisions of the octahedron $T(24\alpha\delta\epsilon, 8\gamma^3, 6\beta^4)$, with 24 tiles.*
2. *Pentagonal subdivisions of the icosahedron $T(60\alpha\delta\epsilon, 20\gamma^3, 12\beta^5)$, with 60 tiles.*

Proof. Note that the pentagonal subdivision of the tetrahedron is not included, because it is symmetric. We will first apply Lemma 8 to get a new vertex $\beta\cdots$ (excluding $\beta^2\gamma$, $\beta\gamma^2$), then apply the irrational angle lemma to control the AVC, and finally try to get all tilings by the AAD. We summarize such computations and deductions in Table 28, excluding the symmetric ($\beta=\gamma$) case $\{\alpha\delta\epsilon, \gamma^3, \beta^3\}$. The proof is completed by Table 28 and 29, similar to Proposition 1.

Table 28: $\alpha\delta\epsilon, \gamma^3$ are vertices.

Vertex		Angles	Irrational Angle Lemma	Contradiction
$\alpha\delta\epsilon, \gamma^3$	$\alpha^2\beta$	$(\frac{5}{6}-\frac{2}{f}, \frac{1}{3}+\frac{4}{f}, \frac{2}{3}, \delta, \frac{7}{6}-\delta+\frac{2}{f})$	$n_4 = n_5$	See Table 29
	$\alpha\beta^2$	$(\frac{4}{3}-\frac{8}{f}, \frac{1}{3}+\frac{4}{f}, \frac{2}{3}, \delta, \frac{2}{3}-\delta+\frac{8}{f})$		
	$\alpha\beta\gamma$	$(1-\frac{4}{f}, \frac{1}{3}+\frac{4}{f}, \frac{2}{3}, \delta, 1-\delta+\frac{4}{f})$		
	$\alpha\beta^3$	$(1-\frac{12}{f}, \frac{1}{3}+\frac{4}{f}, \frac{2}{3}, \delta, 1-\delta+\frac{12}{f})$		
	$\beta^3\gamma$	$(2-\delta-\epsilon, \frac{4}{6}, \frac{2}{3}, \delta, \epsilon), f=36$		Similar to Proposition 5 Case $\{\beta\delta\epsilon, \alpha^2\gamma, \alpha\gamma^3\}$
	β^4	$(2-\delta-\epsilon, \frac{1}{2}, \frac{2}{3}, \delta, \epsilon), f=24$		Admit the unique pentagonal subdivision tiling
	β^5	$(2-\delta-\epsilon, \frac{2}{5}, \frac{2}{3}, \delta, \epsilon), f=60$		

Table 29: AVC for Cases $\{\alpha\delta\epsilon, \gamma^3, \alpha^2\beta\}$, $\{\alpha\delta\epsilon, \gamma^3, \alpha\beta^2\}$, $\{\alpha\delta\epsilon, \gamma^3, \alpha\beta\gamma\}$, and $\{\alpha\delta\epsilon, \gamma^3, \alpha\beta^3\}$.

AVC for $\{\alpha\delta\epsilon, \gamma^3, \alpha^2\beta\}$		AVC for $\{\alpha\delta\epsilon, \gamma^3, \alpha\beta^2\}$		AVC for $\{\alpha\delta\epsilon, \gamma^3, \alpha\beta\gamma\}$		AVC for $\{\alpha\delta\epsilon, \gamma^3, \alpha\beta^3\}$	
f	vertex	f	vertex	f	vertex	f	vertex
all	$\alpha\delta\epsilon, \gamma^3, \alpha^2\beta$	all	$\alpha\delta\epsilon, \gamma^3, \alpha\beta^2$	all	$\alpha\delta\epsilon, \gamma^3, \alpha\beta\gamma$	all	$\alpha\delta\epsilon, \gamma^3, \alpha\beta^3$
60	$\alpha\beta^3, \beta^2\delta\epsilon, \beta^5$	24	$\beta^4, \beta^2\delta\epsilon, \delta^2\epsilon^2$	36	$\beta^3\gamma, \beta^2\delta\epsilon$	60	$\alpha^2\beta, \beta^2\delta\epsilon, \beta^5$
		36	$\beta^3\gamma, \beta\gamma\delta\epsilon$				
		60	$\beta^5, \beta^3\delta\epsilon, \beta\delta^2\epsilon^2$				

By the balance lemma $\beta^k\delta\epsilon, \beta\gamma\delta\epsilon$ or $\beta^k\delta^2\epsilon^2$ must appear. Its AAD gives $\alpha|\alpha$, $\alpha|\gamma$, $\beta|\gamma$, $\gamma|\delta$ or $\delta|\epsilon$, contradicting the AVC

Case. $\{\alpha\delta\epsilon, \gamma^3, \beta^4\}$

The angle sums of $\alpha\delta\epsilon, \gamma^3, \beta^4$ and the angle sum for the pentagon imply $\beta = \frac{1}{2} < \gamma = \frac{2}{3}$. By Lemma 10, we have $\delta > \epsilon$.

Subcase. $\delta \geq 1$

By $R(\delta\epsilon\cdots) = \alpha < 2\beta$, $\beta+\gamma$, 2γ and the parity lemma, we get $\delta\cdots = \delta\epsilon\cdots = \alpha\delta\epsilon$. Therefore AVC $= \{\alpha\delta\epsilon, \gamma^3, \beta^4\}$.

Subcase. $\delta < 1$, $\alpha \geq 1$

By the angle values, we get $\alpha \cdots = \alpha\delta\epsilon$, $\alpha\beta^2$ or $\alpha\epsilon^x (x \geq 4)$. By the previous discussion, we get that $\alpha\beta^2$ is not a vertex. Then we know $\alpha \cdots = \alpha\delta\epsilon$ or $\alpha\epsilon^x (x \geq 4)$. By the balance lemma, we have $\alpha \cdots = \alpha\delta\epsilon$. Therefore AVC $= \{\alpha\delta\epsilon, \gamma^3, \beta^4\}$.

Subcase. $\delta < 1$, $\alpha < 1$

By $\alpha < 1$, we get $\delta + \epsilon > 1$. Then by $R(\delta^2 \cdots) < \alpha, 2\beta, \beta+\gamma, 2\gamma, 2\delta, \delta+\epsilon, 2\epsilon$ and the parity lemma, we get that $\delta^2 \cdots$ is not a vertex. By $R(\delta\epsilon \cdots) = \alpha < 2\beta$, $\beta + \gamma$, 2γ and the parity lemma, we get $\delta \cdots = \delta\epsilon \cdots = \alpha\delta\epsilon$. Therefore AVC $= \{\alpha\delta\epsilon, \gamma^3, \beta^4\}$.

In conclusion, we always have AVC $= \{\alpha\delta\epsilon, \gamma^3, \beta^4\}$.

Similarly, for Case $\{\alpha\delta\epsilon, \gamma^3, \beta^5\}$, we get AVC $= \{\alpha\delta\epsilon, \gamma^3, \beta^5\}$. Therefore, we can easily obtain pentagonal subdivisions of the octahedron and icosahedron (see Figure 4 for 3D pictures, see Table 1 for geometric data). Three pentagonal subdivisions and their moduli spaces were described in [8].

□

Proposition 11. *There is no non-symmetric a^4b-tiling with general angles, such that $\alpha\delta\epsilon$ is the unique a^2b-vertex type and α^3, $\alpha^2\gamma$ or $\alpha\gamma^2$ is a vertex.*

Proof. For each case, we will first apply Lemma 8 to get a new vertex $\beta \cdots$ (excluding $\beta^2\gamma$, $\beta\gamma^2$), then apply the irrational angle lemma to control the AVC. We summarize such computations and deductions in Table 30, excluding the symmetric ($\beta = \gamma$) ones: Case $\{\alpha\delta\epsilon, \alpha^2\gamma, \alpha^2\beta\}$, Case $\{\alpha\delta\epsilon, \alpha\gamma^2, \alpha\beta^2\}$ and $\{\alpha\delta\epsilon, \alpha\gamma^2, \alpha\beta\gamma\}$. The proof is completed by Table 30 and 31.

□

Proposition 12. *There is no non-symmetric a^4b-tiling with general angles, such that $\alpha\delta\epsilon$ is the unique a^2b-vertex type and $\alpha\beta\gamma$ is a vertex.*

Proof. By Proposition 9, 10, 11, we may assume that α^3, $\alpha^2\gamma$ ($\alpha^2\beta$), $\alpha\gamma^2$ ($\alpha\beta^2$), $\beta^2\gamma$ ($\beta\gamma^2$), γ^3 (β^3) are not vertices. Then $\alpha\delta\epsilon$, $\alpha\beta\gamma$ are the only degree 3 vertices. By (2.1) and (2.2), $\#\alpha \geq v_3 = 20 + \sum_{k\geq 4}(3k-10)v_k > f = 12 + 2\sum_{k\geq 4}(k-3)v_k$, a contradiction. □

7 Some non-edge-to-edge quadrilateral tilings

All pentagonal prototiles in Theorem 1 and 2 have fixed β, γ values, and may degenerate to quadrilaterals when α, δ, or ϵ equals π as shown in Figure 33.

Then it induces some non-edge-to-edge quadrilateral tilings as follows:

1. Three pentagonal subdivisions of the Platonic solids with 12, 24 and 60 tiles and $\alpha = \pi$ in Figure 34, and two pentagonal subdivisions of the Platonic solids with 24 and 60 tiles and $\delta = \pi$ in Figure 35;

Table 30: All possible cases with $\alpha^3, \alpha^2\gamma$ or $\alpha\gamma^2$.

Vertex		Angles	Irrational Angle Lemma	Contradiction
$\alpha\delta\epsilon, \alpha^3$	$\alpha\beta\gamma$	$(\frac{2}{3}, \beta, \frac{4}{3}-\beta, \delta, \frac{4}{3}-\delta)$, $f=12$ and all vertices have degree 3		No degree ≥ 4 vertex $\delta\epsilon\cdots$ (Similar to Proposition 4)
	$\alpha^2\beta$	$(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}+\frac{4}{f}, \delta, \frac{4}{3}-\delta)$	$n_4 = n_5$	
	$\alpha\beta^2$			
	β^3			
	$\alpha\beta^3$	$(\frac{2}{3}, \frac{4}{9}, \frac{5}{9}+\frac{4}{f}, \delta, \frac{4}{3}-\delta)$		
	$\beta^3\gamma$	$(\frac{2}{3}, \frac{1}{2}-\frac{2}{f}, \frac{1}{2}+\frac{6}{f}, \delta, \frac{4}{3}-\delta)$		
	β^4	$(\frac{2}{3}, \frac{1}{2}, \frac{1}{2}+\frac{4}{f}, \delta, \frac{4}{3}-\delta)$		
	β^5	$(\frac{2}{3}, \frac{2}{5}, \frac{3}{5}+\frac{4}{f}, \delta, \frac{4}{3}-\delta)$		
$\alpha\delta\epsilon, \alpha^2\gamma$	$\alpha\beta^2$	$(\frac{4}{5}-\frac{8}{5f}, \frac{3}{5}+\frac{4}{5f}, \frac{2}{5}+\frac{16}{5f}, \delta, \frac{6}{5}-\delta+\frac{8}{5f})$		
	$\alpha\beta^3$	$(\frac{5}{7}-\frac{12}{7f}, \frac{3}{7}+\frac{4}{7f}, \frac{4}{7}+\frac{24}{7f}, \delta, \frac{9}{7}-\delta+\frac{12}{7f})$		
	$\beta^3\gamma$	$(\frac{3}{4}-\frac{3}{f}, \frac{1}{2}-\frac{2}{f}, \frac{1}{2}+\frac{6}{f}, \delta, \frac{5}{4}-\delta+\frac{3}{f})$		
	β^4	$(\frac{3}{4}-\frac{2}{f}, \frac{1}{2}, \frac{1}{2}+\frac{4}{f}, \delta, \frac{5}{4}-\delta+\frac{2}{f})$		
	β^5	$(\frac{7}{10}-\frac{2}{f}, \frac{2}{5}, \frac{3}{5}+\frac{4}{f}, \delta, \frac{13}{10}-\delta+\frac{2}{f})$		
	$\alpha\beta\gamma$	$(1-\frac{4}{f}, 1-\frac{4}{f}, \frac{8}{f}, \delta, 1-\delta+\frac{4}{f})$		See Table 31
	β^3	$(\frac{5}{6}-\frac{2}{f}, \frac{2}{3}, \frac{1}{3}+\frac{4}{f}, \delta, \frac{7}{6}-\delta+\frac{2}{f})$		
$\alpha\delta\epsilon, \alpha\gamma^2$	β^3	$(\frac{4}{3}-\frac{8}{f}, \frac{2}{3}, \frac{1}{3}+\frac{4}{f}, \delta, \frac{2}{3}-\delta+\frac{8}{f})$		
	$\alpha^2\beta$	$(\frac{4}{5}-\frac{8}{5f}, \frac{2}{5}+\frac{16}{5f}, \frac{3}{5}+\frac{4}{5f}, \delta, \frac{6}{5}-\delta+\frac{8}{5f})$		
	$\alpha\beta^3$	$(\frac{4}{5}-\frac{24}{5f}, \frac{2}{5}+\frac{8}{5f}, \frac{3}{5}+\frac{12}{5f}, \delta, \frac{6}{5}-\delta+\frac{24}{5f})$		
	$\beta^3\gamma$	$(1-\frac{12}{f}, \frac{1}{2}-\frac{2}{f}, \frac{1}{2}+\frac{6}{f}, \delta, 1-\delta+\frac{12}{f})$		No degree ≥ 4 vertex $\delta\epsilon\cdots$
	β^4	$(1-\frac{8}{f}, \frac{1}{2}, \frac{1}{2}+\frac{4}{f}, \delta, 1-\delta+\frac{8}{f})$		
	β^5	$(\frac{4}{5}-\frac{8}{f}, \frac{2}{5}, \frac{3}{5}+\frac{4}{f}, \delta, \frac{9}{5}-\delta+\frac{8}{f})$		

Table 31: AVC for Cases $\{\alpha\delta\epsilon, \alpha^2\gamma, \alpha\beta\gamma\}$, $\{\alpha\delta\epsilon, \alpha^2\gamma, \beta^3\}$ and $\{\alpha\delta\epsilon, \alpha\gamma^2, \beta^3\}$.

AVC for $\{\alpha\delta\epsilon, \alpha^2\gamma, \alpha\beta\gamma\}$		AVC for $\{\alpha\delta\epsilon, \alpha^2\gamma, \beta^3\}$		AVC for $\{\alpha\delta\epsilon, \alpha\gamma^2, \beta^3\}$	
f	vertex	f	vertex	f	vertex
all	$\alpha\delta\epsilon, \alpha^2\gamma, \alpha\beta\gamma$	all	$\alpha\delta\epsilon, \alpha^2\gamma, \beta^3$	all	$\alpha\delta\epsilon, \alpha\gamma^2, \beta^3$
$8s+4, s=2,3,\cdots$	$\alpha\gamma^{s+1}, \beta\gamma^{s+1}, \gamma^{2s+1}, \gamma^s\delta\epsilon$	60	$\alpha\gamma^3, \gamma^2\delta\epsilon, \gamma^5$	24	$\gamma^4, \gamma^2\delta\epsilon, \delta^2\epsilon^2$
				36	$\beta\gamma^3, \beta\gamma\delta\epsilon$
				60	$\gamma^5, \gamma^3\delta\epsilon, \gamma\delta^2\epsilon^2$
By the balance lemma, $\gamma^k\delta\epsilon$ must appear. Its AAD gives $\beta\mid\delta$, $\beta\mid\epsilon$, $\delta\mid\epsilon$ or $\epsilon\mid\epsilon$, contradicting the AVC				By the balance lemma, $\gamma^k\delta\epsilon, \gamma^k\delta^2\epsilon^2$ or $\beta\gamma\delta\epsilon$ must appear. Its AAD gives $\alpha\mid\alpha$, $\alpha\mid\beta$, $\beta\mid\gamma$, $\beta\mid\delta$, $\beta\mid\epsilon$ or $\delta\mid\epsilon$, contradicting the AVC	

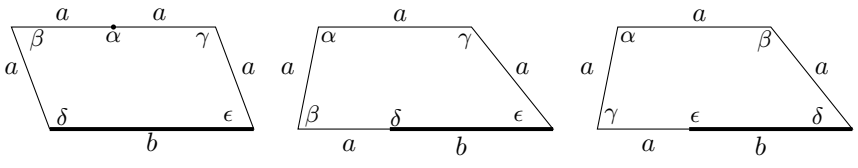


Figure 33: Some quadrilaterals as degenerate pentagons.

2. A sequence of unique pentagons with α or $\epsilon = \pi$ admitting non-symmetric 3-layer earth map tilings $T(4m\,\alpha\delta\epsilon, 2m\,\beta^2\gamma, 2\gamma^m)$ with $4m$ tiles for any $m \geq 4$ (see the first and third pictures of Figure 36), among which each odd $m = 2k+1$ case always admits a standard flip modification: $T((8k+4)\alpha\delta\epsilon, 4k\,\beta^2\gamma, 4\beta\gamma^{k+1})$ (see the second and fourth pictures of Figure 36).

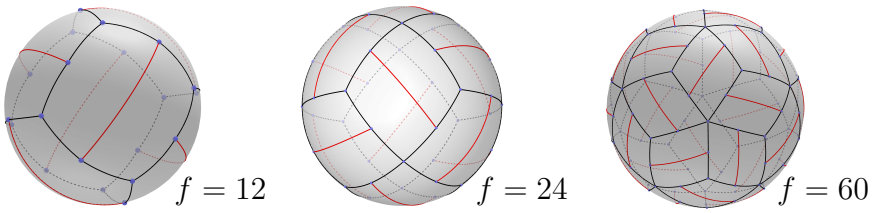


Figure 34: The pentagonal subdivisions of the Platonic solids with $\alpha = \pi$.

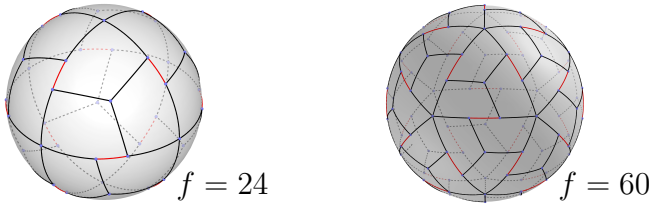


Figure 35: The pentagonal subdivisions of the Platonic solids with $\delta = \pi$.

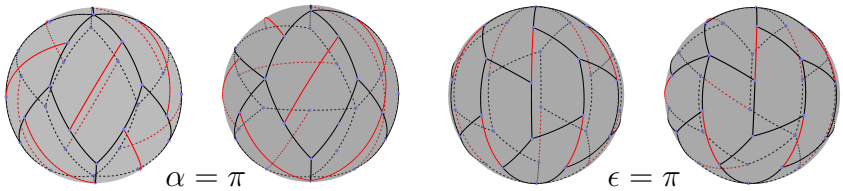


Figure 36: The degenerate non-symmetric 3-layer earth map tilings with 20 tiles and their standard flips.

References

[1] C. Adams. The Tiling Book: An Introduction to the Mathematical Theory of Tilings. American Mathematical Society, Providence, RI, 2022.

[2] C. Adams, C. Edgar, P. Hollander, L. Jacoby. The non-edge-to-edge tilings of the sphere by regular polygons. *Discrete Comput. Geom.*, 72:1029–1085, 2024.

[3] Y. Akama, E. Wang, M. Yan. Tilings of the sphere by congruent pentagons III: edge combination a^5. *Adv. Math.*, 394 (2022), 107881.

[4] H. M. Cheung, H. P. Luk, M. Yan. Tilings of the sphere by congruent quadrilaterals or triangles. *Preprint*, arXiv:2204.02736, 2022.

[5] H. M. Cheung, H. P. Luk, M. Yan. Tilings of the sphere by congruent pentagons IV: edge combination a^4b. *Preprint*, arXiv:2307.11453, 2023.

[6] B. Grünbaum, G. C. Shephard. Tilings and patterns. W. H. Freeman and Dover, 1987 and 2016.

[7] J. Liang, Y. Liao, W.C. Hu, E. Wang. Tilings of the sphere by congruent pentagons V: edge combination a^4b with rational angles. *Preprint*, arXiv:2507.07038, 2025.

[8] J. Liang, E. Wang, M. Yan. Moduli spaces of pentagonal subdivision tilings. *J. Geom. Phys.*, 222 (2026), 105773.

[9] Y. Liao, P. Qian, E. Wang, Y. Xu. Tilings of the sphere by congruent quadrilaterals I: edge combination a^2bc. *Chin. Ann. Math. Ser. B*, 45(5), 2024, 733–766.

[10] Y. Liao, E. Wang. Tilings of the sphere by congruent quadrilaterals II: edge combination a^3b with rational angles. *Nagoya Math. J.*, 253 (2024), 128–163.

[11] Y. Liao, P. Qian, E. Wang, Y. Xu. Tilings of the sphere by congruent quadrilaterals III: edge combination a^3b with general angles. *Forum Math.* 2024; 36(5): 1159–1186.

[12] E. Wang, M. Yan. Tilings of the sphere by congruent pentagons I: edge combinations a^2b^2c and a^3bc. *Adv. Math.*, 394 (2022), 107866.

[13] E. Wang, M. Yan. Tilings of the sphere by congruent pentagons II: edge combination a^3b^2. *Adv. Math.*, 394 (2022), 107867.