Thermodynamics of Kerr-Bertotti-Robinson black hole

Li Hu,[1,2,3,*] Rong-Gen Cai,[4,†] and Shao-Jiang Wang[2,5,‡]

[1] *School of Fundamental Physics and Mathematical Sciences, Hangzhou Institute for Advanced Study (HIAS), University of Chinese Academy of Sciences (UCAS), Hangzhou 310024, China*
[2] *Institute of Theoretical Physics, Chinese Academy of Sciences (CAS), Beijing 100190, China*
[3] *University of Chinese Academy of Sciences (UCAS), Beijing 100049, China*
[4] *Institute of Fundamental Physics and Quantum Technology, & School of Physical Science and Technology, Ningbo University, Ningbo, 315211, China*
[5] *Asia Pacific Center for Theoretical Physics (APCTP), Pohang 37673, Korea*

We investigate the thermodynamic properties of the Kerr-Bertotti-Robinson black hole, an exact Petrov type-D solution of Einstein-Maxwell theory describing a rotating black hole immersed in an external electromagnetic field. While the conserved angular momentum and electric charge can be computed straightforwardly, the conserved mass cannot be obtained through standard integrability methods due to the nontrivial asymptotically uniform external electromagnetic field. To overcome this difficulty, we adopt the Christodoulou-Ruffini mass relation as a thermodynamic definition of the conserved mass and identify the associated generator, thereby fixing the ambiguity in defining this conserved mass and constructing the thermodynamic potentials. These thermodynamic quantities naturally satisfy the first law of black-hole thermodynamics as well as the Smarr formula.

arXiv:2603.18821v3 [gr-qc] 18 Aug 2026

I. INTRODUCTION

Black holes play a central role in both theoretical physics and astronomy. Recent advances, including the direct detection of gravitational waves from binary black hole mergers by LIGO and Virgo [1] and the direct imaging of black-hole shadows by the Event Horizon Telescope [2], have provided compelling evidence for their astrophysical existence. These developments further motivate the study of realistic black-hole environments beyond idealized vacuum solutions [3–6].

Astrophysical black holes, particularly those residing in galactic centers [7, 8], are typically surrounded by accretion disks that can sustain strong magnetic fields [9, 10]. Therefore, constructing exact solutions that incorporate external electromagnetic fields has always been a focus of theoretical research. Early progress in this direction was made by Ernst and Wild [11, 12], who obtained the Kerr-Newman-Melvin spacetime describing a rotating charged black hole immersed in a magnetic universe. However, this solution exhibits several limitations, such as the presence of an ergoregion extending to infinity [13] and the algebraic type-I structure [14], which restrict its applicability in astrophysical contexts.

More recently, Podolský and Ovcharenko have successfully constructed a new exact solution with magnetic field, dubbed the Kerr-Bertotti-Robinson (Kerr-BR) spacetime [15, 16]. This geometry describes a rotating black hole embedded in an external electromagnetic field with improved algebraic structure (Petrov type D) and more tractable properties. As a result, it has attracted considerable attention in recent studies, including investigations of energy extraction via magnetic Penrose processes [17, 18], analyses of black-hole shadows [19–21], extended black hole solutions [22–26], and the influence of external magnetic fields on gravitational waves [27–29], to name just a few. For other studies, see Refs. [30–43].

Despite these developments, the thermodynamic description of the Kerr-BR black hole remains largely unexplored. In this paper, we present the conserved angular momentum J, electric charge Q, and the mass M for the first time. Owing to the nontrivial asymptotic structure, the identification of the appropriate generator associated with the conserved mass is not straightforward. Consequently, the covariant phase space formalism does not directly yield an integrable mass through standard procedures [25, 44–48]. To address this issue, we adopt the Christodoulou-Ruffini mass formula as a thermodynamic definition of the conserved mass [45, 47]. This formula, originally derived from the limit of electric Penrose processes [49, 50], provides a functional dependence $M = M(S, J, Q)$ that implicitly encodes thermodynamic consistency.

Based on this definition, we determine the generator associated with the conserved mass in terms of the black hole mass, spin, and electromagnetic field strength, which turns out to be a nontrivial combination. We then construct the redefined thermodynamic potentials and show that they coincide with those derived directly from the Christodoulou-Ruffini relation. As a consequence, all explicit dependence on the external magnetic field in the first law of black hole thermodynamics and Smarr formula is absorbed into the thermodynamic quantities, so both relations take their standard forms without additional $\mu\delta B$ or μB terms, in agreement with previous results for magnetized black holes [45]. This establishes a consistent thermodynamic description of the Kerr-BR black hole in the presence of external magnetic fields.

* huli21@mails.ucas.ac.cn
† caironggen@nbu.edu.cn
‡ schwang@itp.ac.cn (Corresponding author)

II. KERR-BR BLACK HOLE

The Kerr-BR metric [15, 16] looks explicitly like

$$\mathrm{d}s^2 = \frac{1}{\Sigma^2}\Big[-\frac{\beta}{\rho^2}\big(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi\big)^2 + \frac{\rho^2}{\beta}\mathrm{d}r^2 + \frac{\rho^2}{P}\mathrm{d}\theta^2 + \frac{P}{\rho^2}\sin^2\theta\,\big(a\mathrm{d}t - (r^2+a^2)\mathrm{d}\phi\big)^2\Big], \tag{1}$$

where

$$\rho^2 = r^2 + a^2\,\cos^2\theta\,, \tag{2}$$

$$P = 1 + B^2\Big(m^2\frac{I_2}{I_1^2} - a^2\Big)\cos^2\theta\,, \tag{3}$$

$$\beta = \big(1 + B^2 r^2\big)\Delta\,, \tag{4}$$

$$\Sigma^2 = \big(1 + B^2 r^2\big) - B^2\Delta\cos^2\theta\,, \tag{5}$$

$$\Delta = \Big(1 - B^2 m^2\frac{I_2}{I_1^2}\Big)r^2 - 2m\frac{I_2}{I_1}r + a^2\,, \tag{6}$$

$$I_1 = 1 - \frac{1}{2}B^2a^2\,, \tag{7}$$

$$I_2 = 1 - B^2a^2\,. \tag{8}$$

The three basic parameters m, a, and B stand for the black-hole mass, spin, and electromagnetic field strength, respectively.

The corresponding gauge potential of this asymptotically uniform electromagnetic field can be described as

$$A_\mu \mathrm{d}x^\mu = \frac{\mathrm{e}^{\mathrm{i}\,\gamma}}{2B}\Big[\Sigma,_r\,\frac{a\mathrm{d}t - (r^2+a^2)\mathrm{d}\phi}{r + ia\cos\theta} + (\Sigma - 1)\mathrm{d}\phi + \frac{i\Sigma,_\theta}{\sin\theta}\frac{\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi}{r + ia\cos\theta}\Big], \tag{9}$$

where the parameter γ represents the duality rotation. In case of $m = 0$, $\gamma = 0$ yields a pure magnetic field, and $\gamma = \pi/2$ yields a pure electric field, though γ itself does not influence the strength parameter B [30]. Furthermore, the physical gauge potential is defined as the real counterpart $A_\mu^{\mathrm{real}} \equiv 2\,\mathrm{Re}\,A_\mu$, and the associated electromagnetic field $F_{\mu\nu} = \partial_\mu \mathrm{A}_\nu^{\mathrm{real}} - \partial_\nu \mathrm{A}_\mu^{\mathrm{real}}$ satisfies the Einstein-Maxwell equations, as expected.

Before discussing the thermodynamic properties, several modifications must be made to the Kerr-BR metric and the associated gauge potential. First, we choose an appropriate normalization of the axial Killing vector ∂_ϕ in order to remove possible conical defects along the symmetry axis. The redefined angular coordinate φ is required to have the standard periodicity 2π. Following Ref. [16], this new angle is introduced as $\varphi = \phi\,P_0$, where the normalization factor P_0 is determined by comparing the proper circumference and radius of small circles around the poles $\theta = 0$ and $\theta = \pi$,

$$P_0 = 1 + B^2\Big(m^2\frac{I_2}{I_1^2} - a^2\Big)\,. \tag{10}$$

The second modification concerns the regularity of the gauge potential on the rotation axis. We require that the real component of the azimuthal gauge potential $A_\varphi^{\mathrm{real}}$ vanish at both the north and south poles, ensuring that the gauge potential is regular along the axis. This condition can be satisfied by exploiting the gauge freedom and shifting $A_\varphi^{\mathrm{real}}$ by a real constant A_0, making use of the inherent gauge freedom. A subtle point is that avoiding the Dirac string restricts the duality rotation parameter: one must set $\gamma = 0$, corresponding to a purely magnetic external field. The required constant shift is therefore

$$A_0 = \frac{\sqrt{I_2} - 1}{P_0 B}\,. \tag{11}$$

One may naturally ask whether the magnetic charge U vanishes only after imposing $\gamma = 0$. We have explicitly verified that, for arbitrary values of the duality angle γ, the magnetic charge vanishes identically,

$$U = \frac{1}{8\pi}\int_{\partial C} F_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{1}{4\pi}\int_{\partial C} F_{\theta\varphi}\,\mathrm{d}\theta\,\mathrm{d}\varphi = 0, \tag{12}$$

where C denotes a Cauchy surface. Therefore, the absence of magnetic charge is an intrinsic property of the Kerr-BR solution rather than a consequence of imposing the regularity condition.

After implementing these two modifications, the metric and the gauge potential take the form

$$\mathrm{d}s^2 = \frac{1}{\Sigma^2}\Big[-\frac{\beta}{\rho^2}\big(\mathrm{d}t - \frac{a\sin^2\theta}{P_0}\,\mathrm{d}\varphi\big)^2 + \frac{\rho^2}{\beta}\mathrm{d}r^2 + \frac{\rho^2}{P}\mathrm{d}\theta^2 + \frac{P}{\rho^2}\sin^2\theta\,\big(a\mathrm{d}t - \frac{r^2+a^2}{P_0}\mathrm{d}\varphi\big)^2\Big], \tag{13}$$

and

$$A_\mu^{\mathrm{real}}\mathrm{d}x^\mu = \frac{1}{B}\Big[\mathrm{d}t\frac{1}{\rho^2}\big(ar\,\Sigma,_r + \frac{a\,\Sigma,_\theta}{\tan\theta}\big) + \frac{\Sigma - \sqrt{I_2}}{P_0}\mathrm{d}\varphi - \mathrm{d}\varphi\frac{1}{\rho^2 P_0}\Big(r(r^2+a^2)\Sigma,_r + a^2\sin\theta\cos\theta\,\Sigma,_\theta\Big)\Big]. \tag{14}$$

With these regularity conditions imposed, we can now proceed to compute the thermodynamic parameters defined at the outer horizon.

III. THERMODYNAMICAL QUANTITIES

We now determine the thermodynamic quantities associated with the outer event horizon. The location of the outer horizon is given by the largest root of the function Δ, that is [16],

$$r_+ = \frac{mI_2 + \sqrt{m^2I_2 - a^2I_1^2}}{I_1^2 - B^2m^2I_2}I_1\,, \tag{15}$$

showing that the position of the outer horizon is modified by the presence of the external electromagnetic field B.

The angular velocity of the outer horizon is obtained from the standard relation,

$$\Omega_H = -\left.\frac{g_{t\varphi}}{g_{\varphi\varphi}}\right|_{r=r_+} = \frac{aP_0}{r_+^2+a^2}\,. \tag{16}$$

The Hawking temperature T_H and the Bekenstein-Hawking entropy S follow from the surface gravity κ and the horizon area $\mathcal{A}$, respectively [30, 51, 52],

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{2\pi}\frac{1+B^2r_+^2}{r_+^2+a^2}\Big(m\frac{I_2}{I_1}-\frac{a^2}{r_+}\Big)\,, \tag{17}$$

$$S = \frac{\mathcal{A}}{4} = \frac{\pi}{P_0}\frac{r_+^2+a^2}{1+B^2r_+^2}\,. \tag{18}$$

The Coulomb electrostatic potential is defined with respect to the Killing generator of the horizon $\chi = \partial_t + \Omega_H\partial_\varphi$. Evaluating the gauge potential along this vector at the outer horizon gives rise to the electrostatic potential as

$$\begin{aligned}\Phi_H &= -A_\mu^{\rm real}\chi^\mu|_{r=r_+}\\ &= \frac{-Ba\left[4-4B^2(m^2+a^2)+B^4a^2(4m^2+a^2)\right]}{B^4m^3a^2I_2+mI_1^2(2-3B^2a^2)+2I_1^2\sqrt{m^2I_2-a^2I_1^2}}\\ &\quad\times\frac{m(1-B^2a^2)+\sqrt{m^2I_2-a^2I_1^2}}{4\sqrt{I_2}}\,.\end{aligned} \tag{19}$$

It is worth noting that the electrostatic potential is independent of the polar angle θ, as expected for a regular Killing horizon.

IV. CONSERVED CHARGES

The Kerr-BR metric is derived from the standard Einstein-Maxwell action,

$$S = \frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{-g}\,(R - F_{\mu\nu}F^{\mu\nu})\,. \tag{20}$$

The generalized Killing equations for fields $g_{\mu\nu}$ and A_μ can be expressed as

$$\mathcal{L}_\xi g_{\mu\nu} = 0\,, \tag{21}$$

$$\mathcal{L}_\xi A_\mu + \partial_\mu\lambda = 0\,, \tag{22}$$

where the gauge parameters $\xi = \xi^\mu\partial_\mu$ and λ are a Killing vector field and a real constant, respectively.

According to covariant phase space methods (a formalism pioneered by Iyer-Wald [53] and further developed by Barnich-Brandt [54, 55] to address issues of asymptotic integrability), the total conserved charge $\mathcal{Q}$ depends on the choice of gauge parameters (ξ,λ) and is defined as

$$\mathcal{Q}_{(\xi,\lambda)}[g,A;\bar g,\bar A] = \int_{\partial C}\int_{\bar g}^{g}\int_{\bar A}^{A} k_{(\xi,\lambda)}[\delta g',\delta A';g',A']\,, \tag{23}$$

where C is a Cauchy surface, $\bar g_{\mu\nu}$ and $\bar A_\mu$ are a set of background solutions, and the explicit form of the surface charge 2-forms $k_{(\xi,\lambda)}$ is adopted from Refs. [54, 56],

$$k_{(\xi,\lambda)}[\delta g,\delta A;g,A] = k_\xi^{grav} + k^F_{(\xi,\lambda)}\,. \tag{24}$$

Here,

$$\begin{aligned}k_\xi^{grav} &= \frac{1}{8\pi}(\mathrm{d}^2x)_{\mu\nu}\Big\{\xi^\nu\nabla^\mu h - \xi^\nu\nabla_\rho h^{\mu\rho} + \xi_\rho\nabla^\nu h^{\mu\rho}\\ &\quad + \frac{1}{2}h\nabla^\nu\xi^\mu - h^{\rho\nu}\nabla_\rho\xi^\mu\Big\}\,,\end{aligned} \tag{25a}$$

$$\begin{aligned}k^F_{(\xi,\lambda)} &= \frac{1}{4\pi}(\mathrm{d}^2x)_{\mu\nu}\Big\{-F^{\mu\nu}\delta A_\rho\xi^\rho - 2\xi^\mu F^{\nu\rho}\delta A_\rho\\ &\quad + \left[2h^{\mu\rho}F_\rho{}^\nu - \delta F^{\mu\nu} - \frac{1}{2}hF^{\mu\nu}\right](A_\sigma\xi^\sigma+\lambda)\Big\}\,,\end{aligned} \tag{25b}$$

where we define

$$\begin{aligned}&(\mathrm{d}^2x)_{\mu\nu} = \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}\,\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma\,, \qquad \epsilon_{tr\theta\varphi} = 1\,,\\ &h_{\mu\nu} = \delta g_{\mu\nu}\,, \qquad h^{\mu\nu} = g^{\mu\rho}g^{\nu\sigma}h_{\rho\sigma}\,, \qquad h = g^{\mu\nu}h_{\mu\nu}\,,\\ &\delta F_{\mu\nu} = \partial_\mu\delta A_\nu - \partial_\nu\delta A_\mu\,, \qquad \delta F^{\mu\nu} = g^{\mu\rho}g^{\nu\sigma}\delta F_{\rho\sigma}\,.\end{aligned}$$

Since the integral on the boundary surface ∂C is performed along a path of solutions, the definition is only meaningful if the integrability condition is satisfied [57],

$$\int_{\partial C}\big(\delta_1 k_{(\xi,\lambda)}[\delta_2 g,\delta_2 A;g,A] - \delta_2 k_{(\xi,\lambda)}[\delta_1 g,\delta_1 A;g,A]\big) = 0\,. \tag{26}$$

The conserved angular momentum J and electric charge Q can be obtained straightforwardly by choosing the gauge parameters $(-\partial_\varphi,0)$ and $(0,-1)$, respectively. In these cases, all boundary terms in Eq. (25) vanish identically, rendering the infinitesimal charges δQ and δJ integrable. Their integration yields

$$J = \mathcal{Q}_{(-\partial_\varphi,0)} = \frac{2ma(2-B^2a^2)^3}{(1-B^2a^2)\big[4+B^4a^4+4B^2(m^2-a^2)\big]^2}\,, \tag{27}$$

$$Q = \mathcal{Q}_{(0,-1)} = \frac{2Bma(-2+B^2a^2)}{\sqrt{1-B^2a^2}\big[4+B^4a^4+4B^2(m^2-a^2)\big]}\,. \tag{28}$$

In contrast, obtaining the conserved total energy is considerably more subtle, as the Kerr-BR spacetime is not asymptotically flat, making the appropriate generator for the thermodynamic mass obscure. In general, it is not

as simple as $(\partial_t, 0)$, for which the infinitesimal conserved charge takes the form

$$\not{\delta}\mathcal{Q}_{(\partial_t,0)} = \int_{\partial C} k_{(\partial_t,0)} = c_m \delta m + c_a \delta a + c_B \delta B\,, \quad (29)$$

where

$$c_m = \frac{4I_1\left(4 - 4B^2a^2 - 8B^2m^2 + 8B^4m^2a^2 + B^4a^4\right)}{I_2\left[4 + B^4a^4 + 4B^2(m^2 - a^2)\right]^2}\,, \quad (30)$$

$$c_a = \frac{2B^2ma}{I_2\left[4 + B^4a^4 + 4B^2(m^2 - a^2)\right]^2} \times \left(16 - 8B^2m^2 - 20B^2a^2 + 12B^4m^2a^2 + 8B^4a^4 - B^6a^6\right), \quad (31)$$

$$c_B = \frac{4Bm}{I_2\left[4 + B^4a^4 + 4B^2(m^2 - a^2)\right]^2} \times \left[12(a^2 - m^2) + B^2a^2(14m^2 - 16a^2 + 7B^2a^4 - B^4a^6)\right], \quad (32)$$

and the notation $\not{\delta}$ emphasizes that Eq. (29) does not satisfy the integrability condition.

Given that we have three independent generators, a natural approach is to consider a linear combination, such as $\alpha(\partial_t + \Omega_{int}\partial_\varphi, \Phi_{int})$, to characterize the conserved energy

$$\delta M = \alpha\left(\not{\delta}\mathcal{Q}_{(\partial_t,0)} - \Omega_{int}\delta J - \Phi_{int}\delta Q\right)\,, \quad (33)$$

where α, Ω_{int}, and Φ_{int} are undetermined parameters. Expanding these variations gives the following three equations:

$$\partial_m M = \alpha\left(c_m - \Omega_{int}\partial_m J - \Phi_{int}\partial_m Q\right)\,, \quad (34a)$$
$$\partial_a M = \alpha\left(c_a - \Omega_{int}\partial_a J - \Phi_{int}\partial_a Q\right)\,, \quad (34b)$$
$$\partial_B M = \alpha\left(c_B - \Omega_{int}\partial_B J - \Phi_{int}\partial_B Q\right)\,. \quad (34c)$$

There are three equations, but with four unknowns, indicating that the conserved mass cannot be uniquely determined from integrability conditions alone. Fortunately, previous studies indicate that enforcing thermodynamic consistency leads to a mass that coincides with the well-known Christodoulou-Ruffini mass relation [25, 45–47],

$$M^2(S, J, Q) = \frac{S}{4\pi} + \frac{Q^2}{2} + \frac{\pi(Q^4 + 4J^2)}{4S}\,. \quad (35)$$

Since in Einstein-Maxwell gravity, the intrinsic parameters of a stationary black hole are given by its entropy S, angular momentum J, and electric charge Q, it is natural to adopt this relation as the definition of the conserved mass. In what follows, we determine the corresponding generator of this conserved mass, and the resulting thermodynamic quantities naturally satisfy the first law of black hole thermodynamics and the Smarr formula.

The resulting Christodoulou-Ruffini mass takes the form

$$M(m, a, B) = 2m(2 - B^2a^2) \times \sqrt{\frac{(4 + B^6a^6 - 3B^4a^4 + 4B^4m^2a^2)}{(1 - B^2a^2)\left[4 + B^4a^4 + 4B^2(m^2 - a^2)\right]^3}}\,. \quad (36)$$

This expression possesses several desirable properties. In particular, it reduces to the conserved mass of the Schwarzschild-Bertotti-Robinson spacetime in the non-rotating limit $a \to 0$ [25], and it reproduces the standard Kerr mass when the external magnetic field is switched off $B \to 0$. This provides further support for identifying the above expression as the physical conserved mass.

Besides, substituting this mass into Eq. (34) immediately determines the three previously undetermined parameters in a linear combination of generators,

$$\alpha = \frac{(2 - B^2a^2)^2}{\sqrt{4 + B^4a^4 + 4B^2(m^2 - a^2)}} \times \sqrt{\frac{1 - B^2a^2}{4 + B^6a^6 - 3B^4a^4 + 4B^4m^2a^2}}\,, \quad (37)$$

$$\Omega_{int} = -B^2a\left[1 + \frac{4B^2m^2}{(2 - B^2a^2)^2}\right]\,, \quad (38)$$

$$\Phi_{int} = \frac{4B^3m^2a}{(2 - B^2a^2)^2\sqrt{1 - B^2a^2}}\,. \quad (39)$$

These parameters also exhibit the expected limits. In the absence of rotation ($a = 0$), they reduce to the Schwarzschild-Bertotti-Robinson case,

$$\alpha = \frac{1}{\sqrt{1 + B^2m^2}}\,, \qquad \Omega_{int} = 0\,, \qquad \Phi_{int} = 0\,. \quad (40)$$

On the other hand, when the external magnetic field vanishes ($B = 0$), all parameters return to the Kerr values,

$$\alpha = 1\,, \qquad \Omega_{int} = 0\,, \qquad \Phi_{int} = 0\,. \quad (41)$$

Furthermore, using the quantities obtained above, we introduce the following redefined thermodynamic potentials,

$$T = \alpha T_H\,, \quad \Omega = \alpha(\Omega_H - \Omega_{int})\,, \quad \Phi = \alpha(\Phi_H - \Phi_{int})\,. \quad (42)$$

These quantities are found to coincide with the thermodynamic potentials derived directly from the Christodoulou-Ruffini mass relation [58],

$$T = \frac{\partial M}{\partial S} = \frac{1}{8\pi M}\left[1 - \frac{\pi^2}{S^2}\left(4J^2 + Q^4\right)\right]\,, \quad (43)$$

$$\Omega = \frac{\partial M}{\partial J} = \frac{\pi J}{MS}\,, \quad (44)$$

$$\Phi = \frac{\partial M}{\partial Q} = \frac{Q}{2MS}\left(S + \pi Q^2\right)\,. \quad (45)$$

With these identifications, the variation of the mass takes the standard form of the first law of black-hole thermodynamics (see Appendix A),

$$\delta M = T\delta S + \Omega\delta J + \Phi\delta Q\,, \tag{46}$$

and consequently the Smarr formula,

$$M = 2TS + 2\Omega J + \Phi Q\,. \tag{47}$$

V. CONCLUSIONS AND DISCUSSIONS

In this work, we have studied the thermodynamic properties of the Kerr-Bertotti-Robinson black hole. Using covariant phase space methods, we showed how the conserved angular momentum J and electric charge Q can be calculated. Owing to the presence of an asymptotically uniform external electromagnetic field, the standard formalism does not yield a well-defined conserved mass. To consistently address this issue, we have adopted the Christodoulou-Ruffini mass relation to resolve the ambiguity in the definition of the conserved mass. Based on this definition, we have determined the corresponding generator and constructed the redefined thermodynamic potentials. As a result, the thermodynamic quantities naturally satisfy the standard form of the first law of black-hole thermodynamics and the associated Smarr formula. This demonstrates that a consistent thermodynamic description can be established even in the presence of nontrivial asymptotic structures.

An interesting feature of our results is that, after identifying the conserved thermodynamic mass, neither the first law nor the Smarr formula contains an additional $\mu\delta B$ or μB term, since all explicit dependence on the external magnetic field is absorbed into the thermodynamic quantities. This is consistent with previous analyses of the Kerr-Newman-Melvin black hole.

Finally, the validity of Christodoulou-Ruffini mass relation is an assumption that need not be valid for Kerr-Bertotti-Robinson black hole *a priori*, and whether the proposed thermodynamics will agree with thermodynamics derived from other prescriptions remains to be seen. It remains an interesting open question whether the same mass can be obtained from alternative approaches, such as the conformal method. We leave this for future work.

ACKNOWLEDGMENTS

We thank Yu-Sen An, Marco Astorino, Shan-Ming Ruan, Anna Tokareva, and Jun-Kun Zhao for helpful discussions. This work is supported by the National Key Research and Development Program of China Grants No. 2021YFC2203004, No. 2021YFA0718304, and No. 2020YFC2201501, the National Natural Science Foundation of China Grants No. 12422502, No. 12547110, No. 12588101, No. 12235019, and No. 12447101.

Appendix A: First law of Kerr-Bertotti-Robinson black hole

In this Appendix, we briefly show how the variation of the conserved mass can lead to the form of the first law.

According to Refs. [45, 57], the surface charge associated with the Killing generator of the black-hole horizon is independent of the choice of the spacelike integration surface. As a result,

$$\int_H k_{(\chi,0)} = \int_{\partial C} k_{(\chi,0)}\,. \tag{A1}$$

where the left-hand side is evaluated on the horizon while the right-hand side is evaluated on a spacelike surface extending to infinity.

Evaluating the charge on the horizon gives the standard result [52, 53],

$$\int_H k_{(\chi,0)} = T_H\delta S + \Phi_H\delta Q\,. \tag{A2}$$

To relate this expression to the conserved quantities defined at infinity, we decompose the gauge parameter linearly as

$$\alpha(\chi,0) = \alpha(\partial_t + \Omega_{int}\partial_\varphi, \Phi_{int}) + \alpha\left((\Omega_H - \Omega_{int})\partial_\varphi, -\Phi_{int}\right)\,. \tag{A3}$$

Since the mapping between the surface charge and the gauge parameter is linear, the corresponding charges can be evaluated separately on the spacelike surfaces at the horizon and at infinity [45],

$$\begin{aligned}\int_H k_{(\alpha\xi,0)} = &\int_{\partial C} k_{\alpha(\partial_t+\Omega_{int}\partial_\varphi,\Phi_{int})} \\ &+ \int_{\partial C} k_{\alpha((\Omega_H-\Omega_{int})\partial_\varphi,-\Phi_{int})}\,.\end{aligned} \tag{A4}$$

Using the definitions of the conserved charges

$$\delta Q = \int_{\partial C} k_{(0,-1)}\,, \tag{A5}$$

$$\delta J = \int_{\partial C} k_{(-\partial_\varphi,0)}\,, \tag{A6}$$

$$\delta M = \int_{\partial C} k_{\alpha(\partial_t+\Omega_{int}\partial_\varphi,\Phi_{int})}\,, \tag{A7}$$

one finds

$$\delta M = \alpha\Big(T_H\delta S + (\Omega_H - \Omega_{int})\delta J + (\Phi_H - \Phi_{int})\delta Q\Big)\,. \tag{A8}$$

With the redefined thermodynamic parameters T, Ω, and Φ, the above first law can be written in the standard form,

$$\delta M = T\delta S + \Omega\delta J + \Phi\delta Q\,. \tag{A9}$$

[1] B. P. Abbott et al. (LIGO Scientific, Virgo), "Observation of Gravitational Waves from a Binary Black Hole Merger," Phys. Rev. Lett. **116**, 061102 (2016), arXiv:1602.03837 [gr-qc].

[2] Kazunori Akiyama et al. (Event Horizon Telescope), "First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole," Astrophys. J. Lett. **875**, L1 (2019), arXiv:1906.11238 [astro-ph.GA].

[3] Li Hu, Rong-Gen Cai, and Shao-Jiang Wang, "Distinctive GWBs from eccentric inspiraling SMBH binaries with a DM spike," JCAP **02**, 067 (2025), arXiv:2312.14041 [gr-qc].

[4] Li Hu, Rong-Gen Cai, and Shao-Jiang Wang, "Dynamical friction can flip the hierarchical three-body system," JCAP **08**, 010 (2025), arXiv:2411.14047 [gr-qc].

[5] Rodrigo Vicente, Theophanes K. Karydas, and Gianfranco Bertone, "Fully Relativistic Treatment of Extreme Mass-Ratio Inspirals in Collisionless Environments," Phys. Rev. Lett. **135**, 211401 (2025), arXiv:2505.09715 [gr-qc].

[6] Zhen-Hong Lyu, Rong-Gen Cai, Zong-Kuan Guo, Jian-Feng He, and Jing Liu, "Ring formation from black hole superradiance through repeated particle production on bound orbits," Phys. Rev. D **112**, 104066 (2025), arXiv:2507.03490 [gr-qc].

[7] D. Richstone et al., "Supermassive black holes and the evolution of galaxies," Nature **395**, A14–A19 (1998), arXiv:astro-ph/9810378.

[8] John Kormendy and Luis C. Ho, "Coevolution (Or Not) of Supermassive Black Holes and Host Galaxies," Ann. Rev. Astron. Astrophys. **51**, 511–653 (2013), arXiv:1304.7762 [astro-ph.CO].

[9] R. P. Eatough et al., "A strong magnetic field around the supermassive black hole at the centre of the Galaxy," Nature **501**, 391–394 (2013), arXiv:1308.3147 [astro-ph.GA].

[10] Bei You et al., "Observations of a black hole x-ray binary indicate formation of a magnetically arrested disk," Science **381**, abo4504 (2023), arXiv:2309.00200 [astro-ph.HE].

[11] Frederick J. Ernst and Walter J. Wild, "Kerr black holes in a magnetic universe," J. Math. Phys. **17**, 182 (1976).

[12] Frederick J. Ernst, "Black holes in a magnetic universe," J. Math. Phys. **17**, 54–56 (1976).

[13] G. W. Gibbons, A. H. Mujtaba, and C. N. Pope, "Ergoregions in Magnetised Black Hole Spacetimes," Class. Quant. Grav. **30**, 125008 (2013), arXiv:1301.3927 [gr-qc].

[14] Vojtech Pravda and O. B. Zaslavskii, "Curvature tensors on distorted Killing horizons and their algebraic classification," Class. Quant. Grav. **22**, 5053–5072 (2005), arXiv:gr-qc/0510095.

[15] Hryhorii Ovcharenko and Jiří Podolský, "New class of rotating charged black holes with nonaligned electromagnetic field," Phys. Rev. D **112**, 064076 (2025), arXiv:2508.04850 [gr-qc].

[16] Jiri Podolsky and Hryhorii Ovcharenko, "Kerr Black Hole in a Uniform Bertotti-Robinson Magnetic Field: An Exact Solution," Phys. Rev. Lett. **135**, 181401 (2025), arXiv:2507.05199 [gr-qc].

[17] Xiao-Xiong Zeng and Ke Wang, "Energy extraction from the Kerr-Bertotti-Robinson black hole via magnetic reconnection in a circular and a plunging plasma," Phys. Rev. D **112**, 064032 (2025), arXiv:2507.21777 [gr-qc].

[18] Mirjavoxir Mirkhaydarov, Tursunali Xamidov, Pankaj Sheoran, Sanjar Shaymatov, and Hemwati Nandan, "Non-Monotonic Enhancement of the Magnetic Penrose Process in Kerr-Bertotti-Robinson Spacetime and its Implication for Electron Acceleration," (2026), arXiv:2601.09919 [gr-qc].

[19] Xinyu Wang, Yehui Hou, Xi Wan, Minyong Guo, and Bin Chen, "Geodesics and shadows in the Kerr-Bertotti-Robinson black hole spacetime," JCAP **02**, 050 (2026), arXiv:2507.22494 [gr-qc].

[20] Heena Ali and Sushant G. Ghosh, "Parameter estimation of Kerr-Bertotti-Robinson black holes using their shadows," JCAP **01**, 018 (2026), arXiv:2508.15862 [gr-qc].

[21] Wentao Liu, Yang Liu, Di Wu, and Yu-Xiao Liu, "Universal framework for horizon-scale tests of gravity with black hole shadows," Phys. Rev. D **114**, L021503 (2026), arXiv:2511.06017 [gr-qc].

[22] Faizuddin Ahmed, İzzet Sakallı, and Ahmad Al-Badawi, "Kerr-Bertotti-Robinson Black Holes Surrounded by a Cloud of Strings," (2025), arXiv:2511.11792 [gr-qc].

[23] Hryhorii Ovcharenko and Jiri Podolsky, "Static black holes in an external uniform electromagnetic field: Reissner-Nordström black holes accelerating in Bertotti-Robinson spacetime," Phys. Rev. D **114**, 024068 (2026), arXiv:2602.15462 [gr-qc].

[24] Ahmad Al-Badawi, Faizuddin Ahmed, and Edilberto O. Silva, "Accelerating Bertotti-Robinson Black Holes in a Uniform Magnetic Field," (2026), arXiv:2603.03494 [gr-qc].

[25] Marco Astorino, "Black holes in the external Bertotti-Robinson-Bonnor-Melvin electromagnetic field," Phys. Rev. D **112**, 104077 (2025), arXiv:2508.12908 [gr-qc].

[26] José Barrientos, Adolfo Cisterna, Amaro Díaz, and Keanu Müller, "From Bertotti–Robinson to Vacuum: New Exact Solutions in General Relativity via Harrison and Inversion Symmetries," (2026), arXiv:2602.17581 [gr-qc].

[27] Xiang-Qian Li, Hao-Peng Yan, and Xiao-Jun Yue, "Gravitational-wave imprints of Kerr–Bertotti–Robinson black holes: frequency blue-shift and waveform dephasing," Eur. Phys. J. C **86**, 176 (2026), arXiv:2512.02921 [gr-qc].

[28] Tursunali Xamidov, Sanjar Shaymatov, Qiang Wu, and Tao Zhu, "Gravitational wave signatures from periodic orbits around a Schwarzschild-Bertotti-Robinson black hole," (2026), arXiv:2602.09453 [gr-qc].

[29] Xulong Yuan and Xiangdong Zhang, "External magnetic field influence on massive binary black hole inspiral gravitational waves and its similarity with environmental effects," (2026), arXiv:2603.05084 [gr-qc].

[30] Haryanto M. Siahaan, "Kerr–Bertotti–Robinson spacetime and the Kerr/CFT correspondence," Nucl. Phys. B **1028**, 117514 (2026), arXiv:2512.12533 [gr-qc].

[31] Xiao-Xiong Zeng, Chen-Yu Yang, and Hao Yu, "Optical characteristics of the Kerr–Bertotti–Robinson black hole," Eur. Phys. J. C **85**, 1242 (2025), arXiv:2508.03020 [gr-qc].

[32] Tower Wang, "Innermost stable circular orbit of Kerr-Bertotti-Robinson black holes and inspirals from it: Ex-

act solutions," (2025), arXiv:2508.04684 [gr-qc].
[33] Amnish Vachher, Arun Kumar, and Sushant G. Ghosh, "The influence of uniform magnetic fields on strong field gravitational lensing by Kerr black holes," JCAP **11**, 021 (2025), arXiv:2508.21100 [gr-qc].
[34] Yu-Kun Zhang and Shao-Wen Wei, "Effects of magnetic fields on spinning test particles orbiting Kerr-Bertotti-Robinson black holes," Phys. Rev. D **113**, 104024 (2026), arXiv:2510.07914 [gr-qc].
[35] Marcello Ortaggio, "Einstein-Maxwell fields as solutions of Einstein gravity coupled to conformally invariant nonlinear electrodynamics," Phys. Rev. D **113**, L081503 (2026), arXiv:2511.13665 [gr-qc].
[36] Finnian Gray, David Kubiznak, Hryhorii Ovcharenko, and Jiri Podolsky, "Hidden symmetries and separability structures of Ovcharenko-Podolský and conformal-to-Carter spacetimes," Phys. Rev. D **113**, 044050 (2026), arXiv:2511.21538 [gr-qc].
[37] Lars Andersson, Finnian Gray, and Marius A. Oancea, "Conserved quantities and integrability for massless spinning particles in general relativity," Phys. Rev. D **113**, 064061 (2026), arXiv:2512.07677 [gr-qc].
[38] Marco Astorino, "Static hairy black hole in 4D general relativity," Phys. Rev. D **113**, 024047 (2026), arXiv:2601.16254 [gr-qc].
[39] José Barrientos, Fabrizio Canfora, Adolfo Cisterna, Keanu Müller, and Anibal Neira, "Melvin–Bonnor and Bertotti–Robinson spacetimes with baryonic charge," Eur. Phys. J. C **86**, 673 (2026), arXiv:2601.19858 [gr-qc].
[40] Chao-Hui Wang, Xiang-Cheng Meng, and Shao-Wen Wei, "Magnetic field effects on spherical orbit in Kerr-Bertotti-Robinson spacetime: constraints from jet precession of M87*," (2026), arXiv:2602.03161 [gr-qc].
[41] G. Mustafa, Orhan Donmez, Dhruba Jyoti Gogoi, Sushant G. Ghosh, Ibrar Hussain, and Chengxun Yuan, "Dynamics, ringdown, and accretion-driven multiple quasi-periodic oscillations of Kerr–Bertotti–Robinson black holes," JCAP **07**, 065 (2026), arXiv:2602.08911 [gr-qc].
[42] Haryanto M. Siahaan, "Meissner Effect in Kerr–Bertotti–Robinson Spacetime," (2026), arXiv:2603.00653 [gr-qc].
[43] Junjie Lu and Xin Wu, "Third type of spacetime with the coexistence of integrability and non-integrability," Eur. Phys. J. C **86**, 256 (2026), arXiv:2603.12674 [gr-qc].
[44] G. W. Gibbons, Yi Pang, and C. N. Pope, "Thermodynamics of magnetized Kerr-Newman black holes," Phys. Rev. D **89**, 044029 (2014), arXiv:1310.3286 [hep-th].
[45] M. Astorino, G. Compère, R. Oliveri, and N. Vandevoorde, "Mass of Kerr-Newman black holes in an external magnetic field," Phys. Rev. D **94**, 024019 (2016), arXiv:1602.08110 [gr-qc].
[46] Marco Astorino, "CFT Duals for Accelerating Black Holes," Phys. Lett. B **760**, 393–405 (2016), arXiv:1605.06131 [hep-th].
[47] Marco Astorino, "Thermodynamics of Regular Accelerating Black Holes," Phys. Rev. D **95**, 064007 (2017), arXiv:1612.04387 [gr-qc].
[48] Yunjiao Gao, Zhenbo Di, and Sijie Gao, "General mass formulas for charged Kerr-AdS black holes," Phys. Scripta **99**, 095022 (2024), arXiv:2304.10290 [gr-qc].
[49] D. Christodoulou and R. Ruffini, "Reversible transformations of a charged black hole," Phys. Rev. D **4**, 3552–3555 (1971).
[50] Li Hu, Rong-Gen Cai, and Shao-Jiang Wang, "Third law of repetitive electric Penrose processes," Phys. Rev. D **113**, L061501 (2026), arXiv:2510.26866 [gr-qc].
[51] Jacob D. Bekenstein, "Black holes and entropy," Phys. Rev. D **7**, 2333–2346 (1973).
[52] James M. Bardeen, B. Carter, and S. W. Hawking, "The Four laws of black hole mechanics," Commun. Math. Phys. **31**, 161–170 (1973).
[53] Vivek Iyer and Robert M. Wald, "Some properties of Noether charge and a proposal for dynamical black hole entropy," Phys. Rev. D **50**, 846–864 (1994), arXiv:gr-qc/9403028.
[54] Glenn Barnich and Friedemann Brandt, "Covariant theory of asymptotic symmetries, conservation laws and central charges," Nucl. Phys. B **633**, 3–82 (2002), arXiv:hep-th/0111246.
[55] Glenn Barnich, "Boundary charges in gauge theories: Using Stokes theorem in the bulk," Class. Quant. Grav. **20**, 3685–3698 (2003), arXiv:hep-th/0301039.
[56] Geoffrey Compere, Keiju Murata, and Tatsuma Nishioka, "Central Charges in Extreme Black Hole/CFT Correspondence," JHEP **05**, 077 (2009), arXiv:0902.1001 [hep-th].
[57] Geoffrey Compere, "An introduction to the mechanics of black holes," in 2nd Modave Summer School in Theoretical Physics (2006) arXiv:gr-qc/0611129.
[58] Marco M. Caldarelli, Guido Cognola, and Dietmar Klemm, "Thermodynamics of Kerr-Newman-AdS black holes and conformal field theories," Class. Quant. Grav. **17**, 399–420 (2000), arXiv:hep-th/9908022.