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Generalised anomalies, QCD_4, and holography

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Abstract: During the last decade, the notion of an 't Hooft anomaly has been generalised to the case of discrete symmetries. An interesting instance, discussed by Tanizaki, is the mixed anomaly between the discrete axial symmetry and the flavour and baryonic symmetries in massless QCD_4. The goal of this note is to provide a derivation of this anomaly from a top-down holographic dual of QCD_4. It is found that the topological couplings in the bulk supergravity dual of the D4-D8 system encode Tanizaki's anomaly, once fluctuations around the bulk gauge fields are turned on. A technical challenge for this computation is the difficulty in maintaining gauge invariance of supergravity theories in the presence of D-branes. To overcome this issue, a compact formulation of the flux sector of (massive) type IIA supergravity in the presence of D8 branes is presented. A crucial ingredient in the analysis is the need for smearing the D8 branes, in order to impose τ-shift invariance on the fluctuation ansatz.

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1 Introduction

In quantum chromodynamics (QCD), the global 0-form symmetry acting faithfully on the Hilbert space is a quotient of the naive global symmetry G_{naive} acting on the quarks, by a subgroup $\mathcal{Z}$ of the centre of the gauge group G_{gauge}. This means that in the presence of a background for the global symmetry, the structure group G of the total gauge-global bundle takes the form

$$G = \frac{G_{\text{gauge}} \times G_{\text{naive}}}{\mathcal{Z}} \,. \tag{1.1}$$

Consequently, when turning on a background for the 0-form symmetry, one needs to include additional data specifying the correlation of the gauge and global bundles. Often this can be achieved by introducing a 2-form gauge field. The presence of this additional structure can sometimes lead to new 't Hooft anomalies [1–7]. The case of interest for the present paper is four-dimensional QCD with $SU(N)$ gauge group, and N_f flavours of massless Dirac fermions

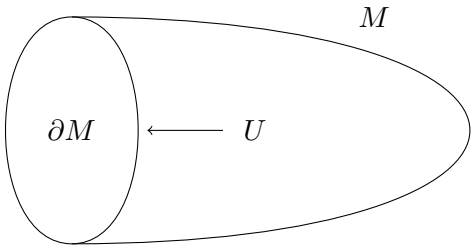


Figure 1. Schematic of a holographic spacetime M. The QFT degrees of freedom live at the boundary ∂M, while the holographic coordinate U extends into the bulk spacetime M. In this setting it is natural to identify the bulk spacetime of the anomaly TFT with the holographic spacetime M.

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in the fundamental representation of $SU(N)$. This theory shall henceforth be referred to as QCD_4. The relevant structure group for the global symmetry bundle in this theory is (2.5). In an interesting paper, Tanizaki demonstrated the presence of a mixed anomaly between the discrete axial symmetry and the flavour symmetry in QCD_4 [4].[1] In the presence of a mass term for the quarks preserving the diagonal component of the chiral symmetry, there is a mixed anomaly between time-reversal and the flavour symmetry when $\theta = \pi$ [1, 3]. While the main focus of this paper is the massless case, brief comments about the massive case appear in section 4. The presence of these novel types of discrete anomalies in QCD_4 pose a natural question as to their holographic origin. The main motivation for this study is to identify such a holographic interpretation. For field theories with a holographic dual in ten or eleven dimensional supergravity, the anomaly is contained in the bulk topological couplings [8]. Intuitively, topological terms in supergravity actions are gauge invariant only on closed manifolds. Thus in holographic contexts where the background has a boundary, their gauge invariance is violated by a boundary term, reproducing the anomaly in the dual field theory. Additionally, the topological sector of the bulk supergravity contains BF-type couplings which encode different global structures of the gauge group in the dual gauge theory, as first noted long ago [9, 10]. The discovery of more subtle anomalies in field theory, such as those alluded to in the previous paragraphs, has recently prompted renewed interest in more careful investigation of these topological couplings for instance in [11–23].

Given a background of the form $M \times L$, which is a product of an external manifold M, and an internal space L, one needs to truncate the bulk topological action, by reducing on the internal space L. The resulting theory, is a topological field theory (TFT) living only on M, whose action contains the anomalies of the dual field theory, see figure 1 for a schematic representation. The TFT in question also contains the BF-type interactions, which once equipped with suitable boundary conditions, can provide a holographic description of the correlation between gauge and flavour bundles discussed in (1.1).

The goal of the present paper is to investigate the topological couplings of the holographic dual of QCD_4. This is achieved by working with the top down holographic QCD proposed in [24] based on the D4-D8-$\overline{\text{D8}}$ brane system, reviewed in appendix A. That the chiral

[1] Throughout this paper, *flavour* symmetry refers to the vector-like subgroup of the non-abelian chiral symmetry of QCD_4, and the abelian component of the vector-like symmetry is referred to as *baryonic* symmetry. This is slightly different from the conventions used in [4].

anomaly of QCD_4 is encoded in the Chern-Simons couplings of the probe D8 branes, was already demonstrated in [24]. The point of the current study is to generalise that discussion by turning on fluctuations for all the bulk supergravity gauge fields, including both RR and NS-NS sectors. This turns out to present some technical challenges as discussed in detail in section 3, with supplementary material tucked away in appendix B. Briefly stated, finding a gauge invariant action for the flux sector of type II supergravities in the presence of D-branes, suitable for studying global properties is subtle. One way to overcome this issue is by employing the *dual* formulation of type IIA supergravity [25]. This enables one to write down the TFT that characterizes the anomalies for the system of our interest. Embedded within this TFT are several recently discovered generalised anomalies. These include Tanizaki's mixed axial-flavour and axial-baryon anomalies that show up in the context of massless QCD_4 [4]. In addition, this TFT contains the so-called anomaly in the space of coupling constants, which is a mixed anomaly between the periodicity of the theta parameter, and the vector-like symmetry of massive QCD. This anomaly was recently investigated in several works, including [1–3].

The rest of this paper is organised as follows. Section 2 contains a general discussion of anomalies in field theory, and an overview of old and new results on anomalies of QCD_4. Section 3 contains a discussion of the so-called dual formulation of type IIA supergravity, written in a compact notation which to the authors' best knowledge is original. Appendix B supplements section 3 with a discussion of the standard and democratic formulation of type IIA supergravity, and contains several definitions used throughout section 3. In section 4, the reduction of the bulk topological couplings on the internal space is performed. The effective lower-dimensional TFT is analysed in this section. A general inflow action for QCD_4 is derived from the holographic set-up, which is shown to contain the chiral anomaly, as well as the novel types of anomaly discovered recently in [3, 4]. Appendix A contains a review of holographic QCD. Section 5 contains an outlook for future research directions.

2 Review: anomalies in field theory

Given a d-dimensional QFT on a manifold $\mathcal{M}_d$ with a global symmetry, one can couple the system to a background $\mathcal{B}$.[2] Under background gauge transformation, $\mathcal{B} \to \mathcal{B}^\Lambda$, the partition function need not be invariant and, in general, can pick up a phase

$$Z_{\mathcal{M}_d}[\mathcal{B}^\Lambda] = \exp\left(-2\pi i \int_{\mathcal{M}_d} \mathcal{A}\left(\Lambda, \mathcal{B}\right)\right) Z_{\mathcal{M}_d}[\mathcal{B}]\,. \tag{2.1}$$

When this phase is non-trivial, the theory is said to have an 't Hooft anomaly. One can restore background gauge invariance, by coupling the system to a $(d+1)$-dimensional bulk manifold $\mathcal{N}_{d+1}$, with boundary $\partial\mathcal{N}_{d+1} = \mathcal{M}_d$, supporting degrees of freedom specified by a local Lagrangian $\mathcal{I}_{d+1}(\mathcal{B})$ on $\mathcal{N}_{d+1}$ subject to

$$\delta_\Lambda \mathcal{I}_{d+1}(\mathcal{B}) = \mathrm{d}\,\mathcal{A}(\Lambda, \mathcal{B}) \tag{2.2}$$

[2]The background $\mathcal{B}$ may be that of a discrete or continuous p-form symmetry with p generic. In this work, discrete gauge fields are always modeled by embedding into $U(1)$ continuous gauge fields subject to a constraint.

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such that the combined system

$$\hat{Z}_{\mathcal{N}_{d+1}}[\mathcal{B}] = Z_{\mathcal{M}_d}[\mathcal{B}] \exp\left(2\pi i \int_{\mathcal{N}_{d+1}} \mathcal{I}_{d+1}(\mathcal{B})\right) \tag{2.3}$$

is now invariant under background gauge transformations. This way of thinking gives us a quantitative measure of the anomaly, and the theory specified by the action $\mathcal{S}(\mathcal{B}) = 2\pi \int \mathcal{I}_{d+1}(\mathcal{B})$ is referred to as the *inflow* action. A theory is said to be anomalous, if its gauge invariant partition function requires specifying a bulk manifold $\mathcal{N}_{d+1}$. Hence the anomaly is trivial if the inflow action is trivial modulo $2\pi\mathbb{Z}$ on closed manifolds. To see this, consider a different choice of a bulk manifold $\mathcal{N}'_{d+1}$. Comparison of the partition function evaluated on $\mathcal{N}_{d+1}$ or $\mathcal{N}'_{d+1}$, shows that they are equivalent if the action evaluated on the closed manifold obtained by gluing $\mathcal{N}_{d+1}$, with $\overline{\mathcal{N}'}_{d+1}$, the orientation reversal of $\mathcal{N}'_{d+1}$ along their common boundary is trivial modulo $2\pi\mathbb{Z}$

$$\frac{\hat{Z}_{\mathcal{N}_{d+1}}[\mathcal{B}]}{\hat{Z}_{\mathcal{N}'_{d+1}}[\mathcal{B}]} = \exp\left(2\pi i \int\limits_{\mathcal{N}\cup\overline{\mathcal{N}'}} \mathcal{I}_{d+1}(\mathcal{B})\right) = 1\,, \quad \text{if} \int\limits_{\mathcal{N}\cup\overline{\mathcal{N}'}} \mathcal{I}_{d+1}(\mathcal{B}) \in \mathbb{Z} \tag{2.4}$$

2.1 Anomalies of QCD$_4$

Given the general definition of anomalies reviewed above, it is interesting to ask about the inflow action for QCD$_4$. As the reader might expect, several studies have already addressed this question. The current subsection serves as a review of existing results, phrased in a notation of our choosing, that makes the holographic match in subsequent sections transparent. For the sake of clarity, let us stress once more that, in this paper, QCD$_4$ refers to $SU(N)$ Yang-Mills theory, minimally coupled to N_f flavours of massless Dirac fermions. The latter are composed of N_f left-handed, as well as N_f right-handed two-component Weyl spinors. The Lagrangian for this theory has a global $[U(N_f)_L \times U(N_f)_R] \,/\, \mathbb{Z}_N$ symmetry, where the quotient by $\mathbb{Z}_N$ is due to the fact that a centre gauge transformation must be removed from flavour phase rotations. The fermion measure in QCD$_4$ suffers an ABJ anomaly which means that the quantum theory has a smaller symmetry group given by

$$\frac{SU(N_f)_L \times SU(N_f)_R \times U(1)_V}{\mathbb{Z}_N \times \left(\mathbb{Z}_{N_f}\right)_V}\,. \tag{2.5}$$

The reader is referred to [4], for a detailed analysis and the origin of the quotients in this symmetry group. What follows is a partial survey of known anomalies of QCD$_4$, relevant to the holographic studies in subsequent sections.[3]

2.1.1 Continuous anomaly

The oldest, and most appreciated 't Hooft anomaly of QCD$_4$ is the continuous anomaly which shows up when one couples the theory to $SU(N_f)_L \times SU(N_f)_R$ gauge fields denoted

[3] In the presence of a background metric, there are additional anomalies, whose discussion goes beyond the scope of this paper. See however section 5, for some comments in this regard.

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by (A_L, A_R). In the presence of this background, the fermion path integral measure picks up a non-trivial Jacobian, which may be canceled by a classical phase in 5d

$$\mathcal{S}[A_L, A_R] = 2\pi N \int \left[\mathrm{CS}_5(A_L) - \mathrm{CS}_5(A_R)\right] , \tag{2.6}$$

where CS_5 denotes the Chern-Simons five-form.[4] This anomaly is of the original type that 't Hooft had in mind [26], although the name 't Hooft anomaly nowadays refers to a much broader class of anomalies.

2.1.2 Discrete anomalies

More subtle anomalies may be detected by careful consideration of the discrete quotients appearing in the symmetry group of QCD (2.5). In particular, in [4], Tanizaki was able to show that turning on a background for the $\left(\mathbb{Z}_{N_f}\right)_L \times U(N_f)_V/\mathbb{Z}_N$ subgroup of the symmetry group of QCD_4 leads to additional discrete anomalies. Here $\left(\mathbb{Z}_{N_f}\right)_L \subset U(1)_L$ is the discrete subgroup of $U(1)_L$ which survives the ABJ anomaly. Including the dynamical gauge bundle, the total structure group of gauge and flavour symmetry of QCD_4 in the presence of this background is[5]

$$\frac{SU(N) \times U(N_f)_V}{\mathbb{Z}_N} \times \left(\mathbb{Z}_{N_f}\right)_L \cong \frac{U(N) \times U(N_f)_V}{U(1)_*} \times \left(\mathbb{Z}_{N_f}\right)_L . \tag{2.7}$$

Let us denote the gauge field for $U(N)$ by a, that for $U(N_f)_V$ by $\tilde{A}$, and the gauge field for $\left(\mathbb{Z}_{N_f}\right)_L$ by A_χ. The quotient by $U(1)_*$ may be achieved by introducing a background 2-form B_2 subject to the constraint [27]

$$N B_2 = \mathrm{tr}(f) , \tag{2.8}$$

where $f = \mathrm{d}\, a + 2\pi\, \mathrm{i}\, a^2$ is the field strength for $U(N)$, and assigning the following 1-form gauge transformation properties

$$B_2 \to B_2 + \mathrm{d}\,\zeta , \quad a \to a + \zeta \mathbb{1}_N , \quad \tilde{A} \to \tilde{A} - \zeta \mathbb{1}_{N_f} . \tag{2.9}$$

The kinetic term of the fundamental quarks invariant under the above gauge transformation rules takes the form

$$\bar{\psi}\gamma^\mu \left(\partial_\mu + a_\mu + \tilde{A}_\mu + A_\chi\right) P_L \psi + \bar{\psi}\gamma^\mu \left(\partial_\mu + a_\mu + \tilde{A}_\mu\right) P_R \psi , \tag{2.10}$$

[4] The exterior derivative of $\mathrm{CS}_k(\tilde{A})$ satisfies

$$\mathrm{d}\,\mathrm{CS}_k(\tilde{A}) = \left[\mathrm{Tr}\, \mathrm{e}^{\tilde{F}}\right]_{k+1} ,$$

and the subscript on the right hand side is an instruction to pick the $(k+1)$-form term in the expansion of the exponential. The conventions used here for the normalization of the gauge field are such that

$$\oint \mathrm{Tr}\, \tilde{F} \in \mathbb{Z} .$$

[5] The quotient by $\mathbb{Z}_N$ on the left hand side corresponds to the identification $(g_c, g_f) \sim (\mathrm{e}^{\frac{2\pi\,\mathrm{i}}{N}} g_c, \mathrm{e}^{\frac{-2\pi\,\mathrm{i}}{N}} g_f)$, where $g_c \in SU(N)$, and $g_f \in SU(N_f)$. The quotient by $U(1)_*$ on the right hand side corresponds to the identification $(\hat{g}_c, \hat{g}_f) \sim (\mathrm{e}^{\mathrm{i}\,\alpha_*}\, \hat{g}_c, \mathrm{e}^{-\mathrm{i}\,\alpha_*}\, \hat{g}_f)$, where $\hat{g}_c \in U(N)$, $\hat{g}_f \in U(N_f)$, and $\mathrm{e}^{\mathrm{i}\,\alpha_*} \in U(1)_*$. Under the natural embedding of $SU(N) \hookrightarrow U(N)$, the two groups are isomorphic.

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where $P_{L,R} = \frac{1}{2}(1 \mp \gamma_5)$. The partition function in the presence of this background is anomalous in the sense of (2.1), and the inflow action is given by

$$\mathcal{S}_5\left[A_\chi, \tilde{A}, B_2\right] = \pi N \int A_\chi \wedge \mathrm{Tr}\left[\tilde{F}^2\right] + 2\pi N \int A_\chi \wedge B_2 \wedge \mathrm{Tr}\,\tilde{F} + \mathcal{O}(A_\chi^2)\,. \tag{2.11}$$

That this expression gives rise to a non-trivial anomaly in the sense of (2.4) follows from the fractional flux quantisation conditions for the discrete fields

$$\oint B_2 \in \frac{1}{N}\mathbb{Z}\,, \quad \oint A_\chi \in \frac{1}{N_f}\mathbb{Z} \tag{2.12}$$

The inflow action (2.11) corresponds to a mixed anomaly between the discrete axial symmetry and flavour and baryonic symmetries.

3 Dual formulation of type IIA fluxes

As explained in the introduction, the holographic origin of the sought after anomalies, is the boundary gauge variation of the topological couplings in the bulk. Therefore, the first step in this programme is to find an action governing the topological sector of the bulk theory. This turns out to be a non-trivial problem in the probe brane backgrounds of the type relevant to QCD_4. The following section contains a reformulation of the dual formulation of type IIA supergravity [25], restricting attention to the flux sector. In this framework, duality constraints follow as a consequence of the equations of motion, as opposed to being imposed by hand. One can couple the bulk action to D-branes in a gauge invariant manner. In addition, the topological couplings appear manifestly in the Lagrangian. Consequently, this is a suitable framework for studying anomalies of type IIA backgrounds. In order to keep the discussion pedagogical, the situation in the absence of D8 branes is considered first. Generalisations to Dp branes, in addition to intersecting Dp-Dq brane systems are possible but are not explicitly discussed here. Appendix B contains a supplementary discussion of the standard and democratic formulations of type IIA supergravity and some useful definitions.

3.1 Without D-branes

The starting point is the action originally proposed in [25], where it was dubbed the *dual* formulation. In the following, only the flux sector of the action is needed, so the metric is assumed to be flat, and the dilaton is set to zero. The bosonic part of the action reads

$$\begin{aligned} S = &-\pi \int \left(\sum_{n=0}^{2} |G_{2n}|^2 + |H|^2 + B G_4^2 - B^2 G_2 G_4 + \frac{B^3}{3}\left(G_0 G_4 + G_2^2\right) - \frac{B^4}{4} G_0 G_2 + \frac{B^5}{20} G_0^2 \right) \\ &-2\pi \int \left[\left(G_4 - B G_2 + \frac{1}{2} B^2 G_0 \right) \mathrm{d}V_5 - (G_2 - B G_0)\,\mathrm{d}V_7 + G_0\,\mathrm{d}V_9 \right], \end{aligned} \tag{3.1}$$

where $H = \mathrm{d}\,B$, and we work in units where $2\pi l_s = 1$. Here, the Kalb-Ramond field B is a 2-form, and the degree of the remaining forms is indicated by a subscript. This action is to be varied with respect to the fluxes G_{2n}, the Kalb-Ramond potential B, and

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the Lagrange multipliers $V_{5,7,9}$. It is convenient to rewrite the above action in a different basis, by introducing fluxes,[6]

$$\Phi_{2n} = \left(\mathrm{e}^{-B}\, G\right)_{2n} , \quad n \in \{0,1,2\} , \tag{3.2}$$

where the subscript on the right hand side denotes the restriction to the $2n$-form part of the expansion. In terms of these variables, the action reads

$$\begin{aligned} S = &-\pi \int \left[\sum_{n=0}^{2} \left| \left(e^{B}\Phi\right)_{2n} \right|^2 + |H|^2 + \left(B\Phi_4^2 + B^2\Phi_2\Phi_4 + \frac{B^3}{3}(\Phi_0\Phi_4 + \Phi_2^2) + \frac{B^4}{4}\Phi_0\Phi_2 + \frac{B^5}{20}\Phi_0^2 \right) \right] \\ &-2\pi \int (\Phi_4 \mathrm{d} V_5 - \Phi_2 \mathrm{d} V_7 + \Phi_0 \mathrm{d} V_9) \end{aligned} \tag{3.3}$$

It is convenient to define the polyforms

$$\Phi = \sum_{n=0}^{2} \Phi_{2n} , \quad V = \sum_{n=0}^{2} V_{2n+5} , \tag{3.4}$$

in terms of which the gauge redundancy of the system may be expressed as

$$B \to B + \mathrm{d}\,\zeta , \quad \Phi \to \left(e^{-\mathrm{d}\,\zeta}\Phi\right)_{<} , \quad V \to \left[e^{-\mathrm{d}\,\zeta} V - \zeta\varphi(-\mathrm{d}\,\zeta)\Phi + \mathrm{d}\,\Lambda\right]_{\geq} ; \quad \Lambda = \sum_{n=0}^{2} \Lambda_{2n+4} , \tag{3.5}$$

where $\varphi(\cdot)$ is defined in (B.12), and we denote the truncation of a polyform to forms of degree less than (resp. greater than or equal to) five by a subscript $<$ (resp. $\geq$). The gauge parameters, satisfy quantisation conditions

$$\oint \mathrm{d}\,\zeta \in \mathbb{Z} , \quad \oint \mathrm{d}\,\Lambda \in \mathbb{Z} , \tag{3.6}$$

where $\oint$ is used to denote cycle integrals. It should be stressed that gauge invariance with respect to ζ is only realised once the constraints from the Lagrange multipliers $V_{5,7,9}$ are imposed. The explicit derivation is presented later, in the discussion leading up to (3.20). The equations of motion following from variation with respect to $\Phi_{0,2,4}$ read

$$\star \left(e^{B}\Phi\right)_{<} = \epsilon_0 \left[e^{B}\left(\Phi + \mathrm{d}\,V\right)\right]_{>} , \tag{3.7}$$

where ϵ_0 is defined in (B.14). Acting on this equation with the Hodge star, it may be recast as

$$\star \left[e^{B}\left(\Phi + \mathrm{d}\,V\right)\right] = \epsilon_0 \left[e^{B}\left(\Phi + \mathrm{d}\,V\right)\right] , \tag{3.8}$$

which is equivalent to the duality constraint (B.13) in the democratic formulation if one identifies

$$\mathcal{G} = e^{B}\left(\Phi + \mathrm{d}\,V\right) . \tag{3.9}$$

[6]The notation here differs from what was originally used in [25], in particular $\Phi_{\text{here}} = F_{\text{there}}$, in order to reserve F_{here} to denote the field strength for the D8-brane gauge field.

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Similarly, the Bianchi identity for $\mathrm{d}V$ and the equations of motion for V can be packaged into a single equation

$$\mathrm{d}\left(\Phi + \mathrm{d}V\right) = 0\,, \tag{3.10}$$

which can be seen to be equivalent to the Bianchi identity (B.9) in the democratic formulation. In the democratic formulation, the equations of motion are not independent, they follow from combining the Bianchi identity and the duality constraint. As such, we have just demonstrated that the dual formulation agrees with the democratic formulation at the level of the equations of motion. The main advantage of working in the Φ-basis of (3.3), as opposed to the G-basis of (3.1) is that the former is written in terms of quantised fluxes. In particular the flux quantisation conditions read

$$\oint \mathrm{d}V \in \mathbb{Z}\,, \quad \oint \Phi \in \mathbb{Z}\,, \tag{3.11}$$

where the first condition is a consequence of the gauge transformation of V with respect to Λ in (3.5) and the second condition follows from the path integration over V.

3.1.1 Compact notation

The action (3.3) may be recast in a more compact form, which will prove its worth when deriving the gauge transformation in the presence of D8 branes. It is useful to define a bracket $\langle\cdot,\cdot\rangle_B$

$$\langle\Phi,\Psi\rangle_B = \int_{M_{10}} \epsilon_0 \left(e^B\Phi\right)_< \wedge \left(e^B\Psi\right)_>\,, \tag{3.12}$$

where we have defined two arbitrary polyforms

$$\Phi = \Phi_0 + \Phi_2 + \Phi_4\,, \quad \Psi = \Psi_0 + \Psi_2 + \Psi_4\,. \tag{3.13}$$

We also adopt the convention that a subscript labelled by $<$ (resp. $>$) denotes the truncation to the forms of degree less than (resp. greater than) 5. In particular we have

$$\begin{aligned}
\left(e^B\Phi\right)_< &= \Phi_0 + \Phi_2 + B\Phi_0 + \Phi_4 + B\Phi_2 + \frac{B^2}{2}\Phi_0\,,\\
\left(e^B\Psi\right)_> &= \left(B + \frac{B^2}{2} + \frac{B^3}{3!}\right)\Psi_4 + \left(\frac{B^2}{2} + \frac{B^3}{3!} + \frac{B^4}{4!}\right)\Psi_2 + \left(\frac{B^3}{3!} + \frac{B^4}{4!} + \frac{B^5}{5!}\right)\Psi_0\,.
\end{aligned} \tag{3.14}$$

With the understanding that the integral in (3.12) only picks up the 10-form part, specifying only one subscript $<$ or $>$ and dropping the other is sufficient

$$\langle\Phi,\Psi\rangle_B = \int_{M_{10}} \epsilon_0 \left(e^B\Phi\right)_< \wedge \left(e^B\Psi\right) = \int_{M_{10}} \epsilon_0 \left(e^B\Phi\right) \wedge \left(e^B\Psi\right)_>\,. \tag{3.15}$$

Moreover, we have that by definition

$$\left(e^B\Phi\right)_< = e^B\Phi - \left(e^B\Phi\right)_>\,. \tag{3.16}$$

These facts can be used to arrive at the following useful identities:

1. The bracket (3.12) is symmetric:

$$\langle\Phi,\Psi\rangle_B = \langle\Psi,\Phi\rangle_B\,. \tag{3.17}$$

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2. Shifting the B-field by an arbitrary 2-form $\hat{B}$ while rescaling $\Phi \to \mathrm{e}^{-\hat{B}}\,\Phi$, and $\Psi \to \mathrm{e}^{-\hat{B}}\,\Psi$ leads to a shift of the bracket by

$$\left\langle \left(\mathrm{e}^{-\hat{B}}\,\Phi\right)_<, \left(\mathrm{e}^{-\hat{B}}\,\Psi\right)_< \right\rangle_{B+\hat{B}} = \langle \Phi, \Psi\rangle_B - \langle \Phi, \Psi\rangle_{-\hat{B}}\,. \tag{3.18}$$

The reader is referred to appendix C for the derivation of these identities. The upshot of this somewhat abstract discussion is that we may now recast the action in the dual formulation (3.3) as

$$S = -\pi \int H \wedge \star H + \pi \int \left(\mathrm{e}^{B}\,\Phi\right)_< \wedge \star \left(\mathrm{e}^{B}\,\Phi\right)_< + \pi\,\langle \Phi, \Phi\rangle_B + 2\pi \int \epsilon_0(\Phi) \wedge \mathrm{d}\,V\,. \tag{3.19}$$

Furthermore, the identities (3.17), and (3.18), together with the flux quantisation conditions (3.11) imply that the gauge variation (3.5) of the action is

$$\delta S = -\pi\,\langle \Phi, \Phi\rangle_{-\mathrm{d}\,\zeta} = 0 \bmod 2\pi\mathbb{Z}\,. \tag{3.20}$$

The second equality above follows by assuming that the 10-manifold on which the action (3.19) is formulated is closed, spin, and equipped with additional suitable structure to guarantee the equality. A sufficient, but not necessary set of conditions is that, the following cycle integrals are always integer quantised

$$\oint \exp\left(\mathrm{d}\,\zeta\right) \in \mathbb{Z}\,, \quad \frac{1}{2}\oint \Phi_4^2 \in \mathbb{Z}\,. \tag{3.21}$$

This can be seen by expanding the gauge variation explicitly

$$-\pi\,\langle \Phi, \Phi\rangle_{-\mathrm{d}\zeta} = \pi \int \left(\mathrm{d}\zeta \Phi_4^2 + (\mathrm{d}\zeta)^2 \Phi_2 \Phi_4 + \frac{(\mathrm{d}\zeta)^3}{3}(\Phi_0\Phi_4 + \Phi_2^2) - \frac{(\mathrm{d}\zeta)^4}{4}\Phi_0\Phi_2 + \frac{(\mathrm{d}\zeta)^5}{20}\Phi_0^2 \right). \tag{3.22}$$

Note that the last term is trivial modulo $2\pi\mathbb{Z}$ on closed spin 10-manifolds with vanishing $\hat{A}$-genus, by the index theorem.[7] On the other hand, the background of interest in this paper, together with the ansatz for the Φ_4 flux (4.3), automatically lead to a gauge invariant action.

3.2 Inclusion of D8 branes

Recall from the introduction that, ultimately, our goal is to formulate an action for the topological sector of the D4-D8 holographic dual of QCD_4. Our next goal is therefore to include the coupling of D8-branes to the bulk action (3.3). Guided by the discussion of the democratic formulation in appendix B, we seek to establish an action functional for (massive) type IIA supergravity, restricted to flat configurations of the Kalb-Ramond 2-form $H = 0$.[8] With this assumption, one can locally write

$$B = \mathrm{d}\,B_1\,. \tag{3.23}$$

[7] Since the gravitational anomalies are not of interest here, it is assumed that all Stiefel-Whitney classes, and all (fractional) Pontryagin classes of the tangent bundle are trivial.

[8] An a posteriori justification for this restriction will be provided in section 4. Essentially it boils down to our choice of the global variant of the gauge group being $SU(N)$, since we wish to consider the theory with fundamental quarks.

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Let y denote the coordinate transverse to the D8 brane, while A is the $U(N_f)$ worldvolume gauge field, and its field strength is denoted by $F = \mathrm{d}\,A + 2\pi\,\mathrm{i}\,A^2$. It is convenient to define a new object Ω satisfying,[9]

$$\mathrm{d}\,\Omega = \mathrm{Tr}\exp(-F)\delta(y)\,\mathrm{d}\,y\,. \tag{3.24}$$

Given the flatness hypothesis for B (3.23) an explicit local expression for Ω can be written as

$$\Omega = \mathrm{e}^{-B}\,\tilde{\Omega}(A, B_1)\,, \qquad \tilde{\Omega}(A, B_1) = N_f\Theta(y) - \sum_{n\geq 0}(-1)^n\,\mathrm{CS}_{2n+1}(A - B_1)\delta(y)\,\mathrm{d}\,y\,, \tag{3.25}$$

where $\Theta(y) = \frac{1}{2}\mathrm{sign}(y)$ denotes the step function, and CS_k is the Chern-Simons k-form. At first glance, this parameterisation of Ω seems unnatural — after all $\mathrm{d}\,\Omega$ is independent of the B-field, while Ω has a complicated explicit dependence. This parameterisation is chosen so that Ω transforms covariantly under the ζ gauge transformations as in (3.30). The topological action in the presence of D-branes, can be constructed by requiring its gauge variation to cancel the chiral anomaly residing at the intersection of a D-brane system [28, 29]. Ignoring the DBI term, we propose the following as the action for the combined bulk-brane system

$$S = \pi\int \left(e^B\Phi\right)_< \wedge\star\left(e^B\Phi\right)_< + 2\pi\int \epsilon_0\,(\Phi+\Omega)_<\wedge(\mathrm{d}\,V-\Omega) + \pi\langle\Phi,\Phi\rangle_B + \pi\int\epsilon_0\Omega_<\wedge\Omega_>\,. \tag{3.26}$$

Note that this expression contains products of $\Theta(y)$ and $\delta(y)$, which are ill-defined. In the general case, to remove this ambiguity one has to regulate these functions. However, in this paper, we are only interested in a background of the form $M_6 \times S^4$, and turn on the gauge field along the M_6 direction. In such a situation the would be problematic terms do not appear. Moreover, in the specific background of interest, we impose τ-shift invariance, as explained in section 4 and appendix A, by smearing the D8 brane. The smearing also removes the aforementioned ill-defined terms appearing in this Lagrangian. For the time being, all that is required is to assume the existence of some Ω with the 1-form gauge transformation prescribed in (3.30). The action (3.26) is to be varied with respect to RR fluxes Φ, as well as the Lagrange multipliers V, while the Kalb-Ramond potential B is treated as a background field. The equations of motion for V now impose the constraint

$$\mathrm{d}(\Phi + \Omega_<) = 0\,. \tag{3.27}$$

Comparison with the modified Bianchi identity for the democratic formulation (B.26) suggests the map

$$\mathrm{e}^{-B}\,\tilde{\mathcal{G}} = \Phi + \mathrm{d}\,V - \Omega_>\,. \tag{3.28}$$

Path integration over V also implies the modified quantisation condition

$$\oint_\Sigma (\Phi + \Omega_<) \in \mathbb{Z}\,. \tag{3.29}$$

[9]Note that, in general, one may also include the contribution of the $\hat{A}$-genus to Ω. This is certainly interesting, and is related to gravitational anomalies. However, it goes beyond the scope of the current work.

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The constituents of the action (3.26) transform under the 1-form gauge symmetry of the Kalb-Ramond field as

$$\begin{gathered} B \to B + \mathrm{d}\,\zeta\,, \quad B_1 \to B_1 + \zeta\,, \quad A \to A + \zeta \mathbb{1}_{N_f}\,, \quad \Omega \to e^{-\mathrm{d}\,\zeta}\Omega\,, \\ \Phi \to \left(e^{-\mathrm{d}\,\zeta}\Phi\right)_<\,, \quad V \to e^{-\mathrm{d}\,\zeta}V - [\zeta\varphi(-\mathrm{d}\,\zeta)(\Phi + \Omega_<)]_>\,. \end{gathered} \tag{3.30}$$

Additionally, there are gauge transformations of the Lagrange multipliers

$$V \to V + \mathrm{d}\,\Lambda\,, \tag{3.31}$$

with Λ defined in (3.5). Finally there is the 0-form gauge redundancy of the D8 brane gauge field

$$A \to A^g = gAg^{-1} - \mathrm{i}\,g\,\mathrm{d}\,g^{-1}\,, \quad \Omega \to \Omega + \mathrm{d}\,\alpha(g, A, B_1)\,, \quad V \to V + \alpha(g, A, B_1)_>\,, \tag{3.32}$$

the explicit form of $\alpha(g, A, B_1)$ will not be necessary for the subsequent discussions. The action (3.26), is not quite gauge invariant with respect to these transformations, rather its gauge transformation cancels the fermion anomaly of D-brane intersections [28, 29]. Varying the action (3.26) with respect to Φ leads to

$$\epsilon_0 \star \left(e^B\Phi\right)_< = \left(e^B(\mathrm{d}\,V - \Omega_> + \Phi)\right)_>\,, \tag{3.33}$$

which is equivalent to the duality constraint in the democratic formulation provided one identifies

$$\tilde{\mathcal{G}} = e^B(\Phi + \mathrm{d}\,V - \Omega_>) \tag{3.34}$$

Under the gauge transformations (3.30), the kinetic terms are invariant, while the topological couplings transform as

$$\delta_\zeta S = -\pi\,\langle \Phi + \Omega_<, \Phi + \Omega_< \rangle_{-\mathrm{d}\,\zeta} = 0 \bmod 2\pi\mathbb{Z}\,, \tag{3.35}$$

on any closed spin 10-manifold, equipped with sufficient additional structure, analogous to those which guarantee (3.20). A sufficient set of conditions is obtained by replacing $\Phi_4 \to \Phi_4 + \Omega_4$ in (3.21).

4 Anomaly matching in holographic QCD

This section contains a holographic derivation of the anomaly inflow action for QCD_4. One way to realise this theory holographically is to consider the D4-D8-$\overline{\mathrm{D8}}$ brane system in type IIA string theory, reviewed in appendix A. The holographic description is obtained by backreacting the N D4 branes [30], while treating the N_f D8-$\overline{\mathrm{D8}}$ pair as probes [24]. The resulting background takes the schematic form $\mathbb{R}^{1,3} \times \text{cigar} \times S^4$, where the cigar is a deformed disk parameterised by U and τ in (A.2), and S^4 denotes the round 4-sphere. In addition to the curved geometry, the background contains a four-form flux

$$\int_{S^4} \Phi_4 = N\,. \tag{4.1}$$

The reader is referred to appendix A for a review of this background, including the explicit metric. Let us denote the coordinates on $\mathbb{R}^{1,3}$ by x^μ, with $\mu \in \{0, 1, 2, 3\}$, and use (y, z) defined in (A.6) to denote the coordinates on the cigar. As discussed in (A.7) The probe D8 branes are located at $y = 0$, and span the remaining directions.

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4.1 Truncation of the topological action on S^4

Following the philosophy of [13], the anomaly theory is obtained by discarding the kinetic terms, and considering fluctuations around the background flux (4.1), expanded in a basis of the cohomology of the internal S^4. The bulk dynamics of these fluctuations are governed by type IIA supergravity action. In section 3, an action functional suitable for precisely such a situation was constructed (3.26). This action contains gauge invariant kinetic terms, and topological couplings, which are gauge invariant up-to total derivative terms. Hence the anomaly in the holographic setup is encoded in these topological terms — the anomalous phase of the partition function of the dual field theory is reproduced by the anomalous gauge variation of the topological interactions on the $\mathbb{R}^{1,3} \times S^1$ boundary geometry. Discarding the kinetic terms in (3.26) leads one to

$$S_{\text{top}} = 2\pi \int \epsilon_0 \, (\Phi + \Omega)_< \wedge (\mathrm{d}\, V - \Omega) + \pi \langle \Phi, \Phi \rangle_B + \pi \int \epsilon_0 \Omega_< \wedge \Omega_> \,. \tag{4.2}$$

Consider the following ansatz for the flux sector

$$B_2 = b_2 = \mathrm{d}\, b_1 \,, \quad \Phi = \hat{\phi}_0 \omega_4 + \phi \,, \quad \mathrm{d}\, V = \mathrm{d}\, \hat{v} \omega_4 + \mathrm{d}\, v \,, \tag{4.3}$$

where ω_4 is the unit-normalised volume-form of the S^4

$$\int_{S^4} \omega_4 = 1 \,, \tag{4.4}$$

and a compact notation for the fluxes analogous to section 3 is used

$$\phi = \phi_0 + \phi_2 + \phi_4 \,, \quad \hat{v} = \hat{v}_1 + \hat{v}_3 + \hat{v}_5 \,, \quad v = v_5 + v_7 + v_9 \,. \tag{4.5}$$

Note, in particular, that all the dependence on S^4 is contained in ω_4. That is, the fluctuations $\{b, \hat{\phi}_0, \phi, \hat{v}, v\}$ live purely on the external manifold $M_6 = \mathbb{R}^{1,3} \times \text{cigar}$. This ensures invariance under $SO(5)$ isometry of the S^4, the significance of this will be emphasised momentarily.

Substituting the ansatz (4.3) into the action (4.2), and integrating over the internal S^4, one obtains

$$S_{\text{top}} = -2\pi \int_{M_6} \hat{\phi}_0 (\mathrm{d}\, v_5 - \Omega_6) + 2\pi \int_{M_6} \epsilon_0 (\phi + \Omega)\, \mathrm{d}\, \hat{v} - 2\pi \int_{M_6} \hat{\phi}_0 e^{b_2} \phi \,. \tag{4.6}$$

Following the discussion in the introduction around (2.4), our criterion for detecting anomalies, is to evaluate this topological action on a closed manifold, obtained by gluing together M_6 with the orientation reversal of some other six-manifold M_6' with an $\hat{\mathbb{R}}^{1,3} \times S^1$ boundary, where $\hat{\mathbb{R}}^{1,3}$ denotes the compactification of $\mathbb{R}^{1,3}$. The path integral over v_5 implies that $\hat{\phi}_0$ is constant and normalised to be $\hat{\phi}_0 \in \mathbb{Z}$. In the particular background of interest, the specific value is fixed by (4.1) to be

$$\hat{\phi}_0 = N \,. \tag{4.7}$$

The equations of motion for $\hat{v}$ read

$$\mathrm{d}(\phi + \Omega_<) = 0 \,, \tag{4.8}$$

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whose general solution may be locally written down as

$$\phi = \mathrm{d}\, u + n - \Omega_{<}\,, \tag{4.9}$$

where $n \in \mathbb{Z}$ is an integration constant and $u = u_1 + u_3$. Note that n, u_1, and u_3 are subject to the Dirac quantisation conditions

$$n \in \mathbb{Z}\,, \quad \oint \mathrm{d}\, u_1 \in \mathbb{Z}\,, \quad \oint \mathrm{d}\, u_3 \in \mathbb{Z}\,, \tag{4.10}$$

which follow from path integrating over $\hat{v}_5$, $\hat{v}_3$, and $\hat{v}_1$. The topological action obtained after integrating out $\hat{v}_5$, $\hat{v}_3$, and $\hat{v}_1$ reads

$$S_{\text{top}} = -2\pi N \int_{M_6} \mathrm{e}^{b_2}\, (\mathrm{d}\, u + n - \Omega)\,. \tag{4.11}$$

As explained in appendix A, Witten's background contains symmetries which are not physical symmetries of QCD$_4$, including the $SO(5)$ isometry of the S^4, $U(1)_\tau$ isometry generated by shifts of the τ coordinate, and a $\mathbb{Z}_2^{(\tau)}$ τ-parity [31]. In order to extract physical observables of QCD$_4$, rather than some artifact carried by the KK tower, one has to project onto the singlet sector of the unwanted symmetries. This is achieved for the $SO(5)$ symmetry, due to the fluctuation ansatz in (4.3), as well as the ansatz for the D8-brane gauge field — no component of the gauge fields is turned on along the S^4 direction. Although the D8-brane embedding explicitly breaks the isometries of the τ coordinate, one might still try to restore it by smearing the D8 branes. The reader should note that, at present, this is an assumption of our construction. It is motivated by the fact that the glueball sector of QCD, should be similar to that of pure Yang-Mills theory. It can be shown that the gluons are invariant under $SO(5)$, $U(1)_\tau$ and $\mathbb{Z}_2^{(\tau)}$ and hence the glueballs of pure Yang-Mills should also be invariant under these symmetries as discussed in [31]. Therefore in the following, projection onto $U(1)_\tau$ and $\mathbb{Z}_2^{(\tau)}$ singlet components of all closed string fields is imposed (see appendix A). Imposing τ-parity kills the constant mode n in the ϕ fluctuation. After smearing the D8-brane, (4.11) becomes

$$S_{\text{top}} = -2\pi N \int_{M_6} e^{b_2} \left(\mathrm{d}\, u - \Omega^{(\mathrm{s})}\right), \tag{4.12}$$

where it is understood that the integral picks up only the contribution from the 6-form component of the integrand. Here, $\Omega^{(\mathrm{s})}$ is the smeared version of Ω as indicated in the superscript, and it is defined as

$$\Omega^{(\mathrm{s})} = \mathrm{e}^{-b_2}\, \tilde{\Omega}^{(\mathrm{s})}(A - b_1)\,, \tag{4.13}$$

and $\tilde{\Omega}^{(\mathrm{s})}$ is defined in (A.17). It is possible to expand the topological action (4.12) to get the more explicit form

$$S_{\text{top}} = -2\pi N \int_{M_6} \left(b_2\, \mathrm{d}\, u_3 + \frac{1}{2} b_2^2\, \mathrm{d}\, u_1 - \tilde{\Omega}_6^{(\mathrm{s})}(A, b_1)\right) \tag{4.14}$$

This action is invariant with respect to the 1-form gauge symmetry

$$b_2 \to b_2 + \mathrm{d}\, \zeta_1\,, \quad b_1 \to b_1 + \zeta_1\,, \quad u \to \left[e^{-\mathrm{d}\, \zeta_1} u\right]_{<4}\,, \quad A \to A + \zeta_1 \mathbb{1}_{N_f}\,. \tag{4.15}$$

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Note however that the fluxes in (4.10) are not gauge invariant with respect to the above. This is unavoidable in supergravity [32]. The terms depending on the D8 brane gauge field in the inflow action read

$$S_{\text{top}} \supset -2\pi N \int \tilde{\Omega}_6^{(s)}(A, b_1) = 2\pi N \int \left[\text{CS}_5(A_L - b_1) - \text{CS}_5(A_R - b_1)\right] . \tag{4.16}$$

As explained in section 2, the chiral anomaly can be detected by turning on continuous chiral gauge fields, while imposing the fluctuations of the B-field to vanish. Setting $b = 0$ in (4.16) yields

$$\mathcal{S}(A_L, A_R) = 2\pi N \int \left(\text{CS}_5(A_L) - \text{CS}_5(A_R)\right) , \tag{4.17}$$

in agreement with the field theory expectation (2.6). This anomaly was already matched holographically in [24].

4.2 Discrete anomalies

In the remainder of this section, we aim to see how the discrete anomalies discussed in section 2 emerge from the topological action in (4.12). The first term in the topological action in (4.14) is a BF-term whose role is to specify the global structure of the gauge group of the dual field theory [9, 12]. Given that QCD_4 contains quarks in the fundamental representation, the global structure of the gauge group must be $SU(N)$, rather than a discrete quotient thereof. The corresponding holographic boundary condition is to let u_3 fluctuate on the boundary in (4.14), while fixing the holonomy of b_2 as a $\mathbb{Z}_N$ element. A detailed discussion of the possible topological boundary conditions for this topological action, in the absence of D8-branes appears in [33]. For the purpose of our discussion, the relevance of the BF coupling is to set the b_2 field to be a $\mathbb{Z}_N$ gauge field. To see that, one can simply consider the theory on $M_4 \times S^2$, obtained by compactifying $\mathbb{R}^{1,3}$ to some compact 4-manifold M_4 and gluing the cigar with its orientations reversal. The path integral over u_3 then imposes

$$\mathrm{d}\, b_2 = 0 , \quad \oint b_2 \in \frac{\mathbb{Z}}{N} , \tag{4.18}$$

reproducing the desired flux quantisation for the B-field in field theory (2.12). One could view this as an a posteriori justification of starting with a bulk supergravity action for $H = 0$ configurations in (3.26).

It is useful to consider this action formulated on a compact manifold, as discussed around (A.13). This then enables one to combine the $\mathcal{O}({b_2}^2)$ terms on the D8 brane with the term linear in $\mathrm{d}\, u_1$ in the TFT (4.14), as this combination is gauge invariant

$$2\pi N \int \frac{1}{2} {b_2}^2 \left(\mathrm{d}\, u_1 + \frac{1}{2\pi} \operatorname{Tr}(A_L - A_R)\, \mathrm{d}\, \tau\right) + \cdots . \tag{4.19}$$

The kinetic term, which was mostly ignored until this point, includes a coupling of the form

$$S_{\text{kin}} \supset \pi \int \left|\mathrm{d}\, u_1 + \frac{1}{2\pi} \operatorname{Tr}(A_L - A_R)\, \mathrm{d}\, \tau\right|^2 , \tag{4.20}$$

which is a Stueckelberg mass for the axial component of the gauge field. That only the axial component of the gauge field picks up a Stueckelberg mass is a consequence of imposing

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invariance under $U(1)_\tau$ on the closed string sector fluctuations. At scales well below the Stueckelberg mass, the axial gauge field must satisfy

$$\mathrm{d}\, u_1 + \frac{1}{2\pi}\,\mathrm{Tr}(A_L - A_R)\,\mathrm{d}\,\tau = 0\,. \tag{4.21}$$

One way to argue for this equation is to label $(u_1)_\tau = u_0/(2\pi)$, and define the gauge invariant 1-form

$$\mathcal{A} = \mathrm{d}\, u_0 + \mathrm{Tr}(A_L - A_R) \tag{4.22}$$

Since $\mathcal{A}$ is globally defined, it cannot have topologically non-trivial configurations

$$\oint \mathrm{d}\,\mathcal{A} = 0\,. \tag{4.23}$$

Moreover, it can be continuously deformed to a trivial configuration $\mathcal{A} \to 0$ that dominates at low energies due to the presence of the mass term (4.20). To see this, introduce a continuous parameter $t \in [0,1]$, and construct an extrapolating 1-form $\mathcal{A}^i_t$ on each open patch $\mathcal{U}_i$. On double overlaps $\mathcal{U}_i \cap \mathcal{U}_j$, one has

$$\mathrm{Tr}\left(A^i_{L,R}\right) = \mathrm{Tr}\left(A^j_{L,R}\right) + \mathrm{d}\,\alpha^{ij}_{L,R}\,, \quad u^i_0 = u^j_0 - \left(\alpha^{ij}_L - \alpha^{ij}_R\right) \tag{4.24}$$

Now define a new $U(1)$ gauge field

$$\hat{A}^i_t = \mathrm{Tr}\left(A^i_L - A^i_R\right) + (t-1)\left[\mathrm{Tr}\left(A^i_L - A^i_R\right) + \mathrm{d}\, u^i_0\right]\,, \tag{4.25}$$

where it should be noted that the term inside the square brackets is gauge invariant. The globally defined extrapolating 1-form is defined in terms of this $U(1)$ connection as $\mathcal{A}^i_t = \hat{A}^i_t + \mathrm{d}\, u^i_0$. As $t \to 0$, $\mathcal{A}_t \to 0$, while $\mathcal{A}_{t=1}$ goes back to the original configuration (4.22). It therefore follows that, at low energies, the axial gauge field must satisfy

$$\mathrm{d}\,\mathrm{Tr}(A_L - A_R) = 0\,, \qquad \oint \mathrm{Tr}(A_L - A_R) \in \mathbb{Z}\,. \tag{4.26}$$

It is instructive to rephrase this discussion in terms of the low energy degrees of freedom. In holographic QCD, the boundary value of u_1 is identified with the QCD ϑ parameter [34]

$$\int_{\text{cigar}} \mathrm{d}\, u_1 = \int_{S^1} u_1 = \vartheta\,. \tag{4.27}$$

While the η' meson is identified with the integral of the trace of the axial component of the gauge field[10]

$$\frac{1}{2\pi}\int_{\text{cigar}} \mathrm{Tr}(A_L - A_R)\,\mathrm{d}\,\tau = \eta' \tag{4.28}$$

[10] In the unsmeared description, the integral over the z direction of the trace of the D8 brane gauge field is interpreted as the η' meson [24]

$$\eta' = \int_{\text{cigar}} \mathrm{tr}(A)\delta(y)\,\mathrm{d}\, y$$

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The combination $(\theta - \eta')$ is natural as it is this particular object which is gauge invariant with respect to the gauge symmetry on the D8 brane. It was shown in [24] that the Stueckelberg action (4.20) contains a mass term for the η' meson.

To recover Tanizaki's anomaly, one must restrict to the desired configuration of background fields corresponding to the correct subgroup of the chiral symmetry (2.7)

$$A_L = \tilde{A} + A_\chi \mathbb{1}_{N_f} \,, \qquad A_R = \tilde{A} \,. \tag{4.29}$$

The constraints (4.26), restricted to this background (4.29), are precisely the conditions that characterise A_χ as a $\mathbb{Z}_{N_f}$ gauge field. Substituting this configuration into (4.16), using (A.15), yields

$$2\pi N \int \left(\frac{1}{2} A_\chi \wedge \mathrm{Tr}(\tilde{F}^2) + A_\chi \wedge b_2 \wedge \mathrm{Tr}(\tilde{F}) + \mathcal{O}\left(A_\chi^2\right) \right) . \tag{4.30}$$

This is precisely the anomaly written down by Tanizaki in [4].

4.3 Coupling constant anomaly of massive QCD$_4$

Let us now move on to see how the anomaly of massive QCD discussed for instance in [1, 3] emerges from the TFT (4.14). To do so, one must restrict to the appropriate profile for the background gauge fields of the appropriate symmetry group. In particular we must turn on a diagonal component of the flavour symmetry. In the holographic picture this is achieved by setting

$$A_L = A_R \,. \tag{4.31}$$

With this configuration the topological action (4.14) is reduced to just two terms

$$2\pi N \int_{M_6} \left(b_2 \, \mathrm{d}\, u_3 + \frac{1}{2} b_2^2 \, \mathrm{d}\, u_1 \right) . \tag{4.32}$$

In the smeared description used here, one can integrate over the τ direction to obtain a five-dimensional action[11]

$$2\pi N \int_{M_5} \left(b_2 \, \mathrm{d}\, u_2 + \frac{1}{2} b_2^2 \, \mathrm{d}\, u_0 \right) , \tag{4.33}$$

where we have defined

$$(u_2)_{MN} = \frac{1}{2\pi} (u_3)_{MN\tau} \,, \quad u_0 = \frac{1}{2\pi} (u_1)_\tau \,. \tag{4.34}$$

The action (4.33) matches the inflow action for the anomaly of massive QCD$_4$ reported for instance in [3].

[11] In order to match the anomaly as presented in [3], one needs to use the dictionary $b_2 \to \frac{1}{N} w_2^{(N)}$, together with (4.27).

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5 Conclusions and outlook

This work establishes a refinement of the holographic dictionary between large N QCD_4, and its top-down supergravity dual. The following contains a summary of our outlook for future research directions.

Some immediate generalisations are as follows. It is possible to consider turning on the contribution of the $\hat{A}$-genus to the D8 brane Wess-Zumino term. This allows one to study gravitational anomalies. The holographic dual of $SO(N)$, and $Sp(N)$ QCD_4 were constructed in [35], and an analysis of the topological sector of those backgrounds could illuminate discrete anomalies of those theories.

Similar probe brane backgrounds for QCD_3 based on a D3-D7 brane system [36], and QCD_2 based on a D2-D8 system exist. A holographic study of their generalized anomalies analogous to the present work is desirable. In the case of QCD_3, a detailed field theoretic study of the anomalies is available in [6].

One of the main results of this work, is the action for the flux sector of type IIA supergravity discussed in section 3. It would be interesting to consider other IIA supergravity backgrounds with known field theory duals using this dual formulation.

Recently, a proposal for the holographic dual of $U(N)$ Yang-Mills has been refined in [18, 37], building on earlier results [8, 10, 38]. It would be interesting to build on this literature and gain an understanding of the holographic dual of $U(N)$ QCD_4.

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A Review of top-down holographic QCD

This appendix contains a review of the main features of the top down holographic QCD used in the main text. While it contains sufficient material for the reader to follow the analysis in the main text, more detail can be found in [24]. The type IIA brane system which realises QCD_4 can be summarised in the following table

	0	1	2	3	(4)	5	6	7	8	9
D4	×	×	×	×	×					
D8-$\overline{\text{D8}}$	×	×	×	×		×	×	×	×	×

. (A.1)

The worldvolume theory of the D4 branes is maximally supersymmetric Yang-Mills theory in five dimensions, compactified on a thermal circle along the x^4 direction. Due to anti-periodic

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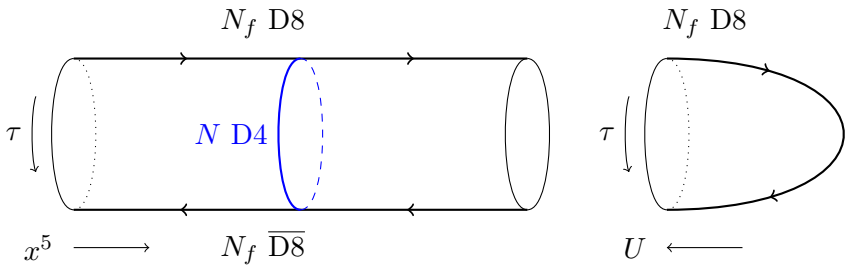


Figure 2. Pictorial representation of the brane system for QCD_4, and its holographic dual. (a) Before backreaction the chiral symmetry is realised as the gauge symmetry on the separate stacks of N_f D8, and $\overline{\text{D8}}$ branes. (b) After backreaction of the N D4 branes, the geometry ends at the tip of the cigar, where the N_f D8 and $\overline{\text{D8}}$ branes meet to form a single continuous stack of D8 branes. This phenomenon is interpreted as the holographic manifestation of chiral symmetry breaking.

boundary conditions along x^4 for the fermions, a tree level mass is generated for the lowest fermionic Kaluza-Klein mode of the gaugino. Furthermore, a 1-loop mass is generated for the scalar superpartners of the gauge field. At high temperatures the theory on the worldvolume is therefore effectively pure Yang-Mills theory in four dimensions. The massless modes of the open strings stretching between the D4 branes and the D8 (resp. $\overline{\text{D8}}$) branes, are left-handed (resp. right-handed) Weyl fermions in the fundamental representation of the SU(N) gauge group. Thus, the massless modes of this brane system are precisely those of QCD_4. Figure 2 depicts a pictorial representation of this brane system. In order to obtain a holographic description, one replaces the N D4 branes by Witten's solution [30], while treating the D8-$\overline{\text{D8}}$ pair as a probe. This description is valid so long as we keep $N \gg N_f$, i.e. the 't Hooft limit of QCD. Witten's background, is given by [30]

$$\begin{aligned} ds^2 &= \left(\frac{U}{R}\right)^{\frac{3}{2}} \left(\eta_{\mu\nu} dx^\mu dx^\nu + f(U) d\tau^2\right) + \left(\frac{R}{U}\right)^{\frac{3}{2}} \left(\frac{dU^2}{f(U)} + U^2 d\Omega_4^2\right) , \\ e^\phi &= g_s \left(\frac{U}{R}\right)^{\frac{3}{4}} , \quad \int_{S^4} \mathrm{d}\, C_3 = N , \quad f(U) = 1 - \left(\frac{U_{KK}}{U}\right)^3 , \end{aligned} \tag{A.2}$$

where x^μ, and τ parameterise the worldvolume directions of the D4 brane, while $d\Omega_4$ is the line element of S^4. The radial holographic coordinate is denoted as U, it is bounded from below as $U \geq U_{\text{KK}}$, R is a constant. The τ coordinate is periodic, with periodicity

$$\tau \sim \tau + \frac{2\pi}{M_{KK}} , \tag{A.3}$$

where M_{KK} is the Kaluza-Klein mass, related to the parameters of the solution by

$$M_{KK} = \frac{3}{2} \frac{U_{KK}^{\frac{1}{2}}}{R^{\frac{3}{2}}} . \tag{A.4}$$

In what follows, we will always work in units where $M_{KK} = 1$.

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A.1 The probe D8 branes

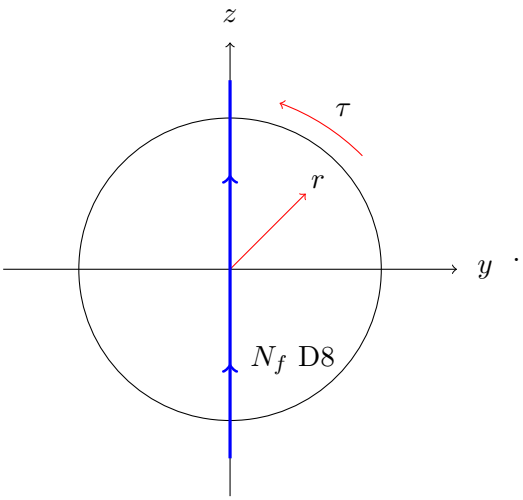
In order to incorporate massless quarks into the background (A.2), one introduces N_f D8 branes treated as probes, that is to neglect their backreaction on the geometry. This approximation is valid, provided N_f is an $\mathcal{O}(1)$ number, in the 't Hooft limit. To describe the D8 brane embedding, it is convenient to perform a change of coordinates

$$U^3 = U_{KK}^3 \left(1 + r^2\right) , \tag{A.5}$$

and further define

$$y = r\cos\tau , \quad z = r\sin\tau , \tag{A.6}$$

such that (U, τ) directions are mapped to (y, z). Viewed in the (y, z)-plane, the D8-brane embedding takes a rather simple form; the N_f D8 branes are extended along the z-direction, and located at $y = 0$ as illustrated below



(A.7)

A crucial feature of this embedding is that it gives a geometric manifestation of chiral symmetry breaking in QCD. Prior to backreaction, the chiral fermions resided on the separate stacks of N_f D8, and N_f $\overline{\text{D8}}$ branes in figure 2. However, due to the curvature of the backreacted geometry, the two stacks meet and reconnect at the tip of the cigar — or the origin of the (y, z)-plane — to form a single stack of N_f D8 branes.

The worldvolume gauge field on the D8 brane is identified with the background gauge field for the internal global symmetry of QCD. However, not all components of this gauge field are physical. The background (A.2) has an SO(5) isometry of the internal S^4. Clearly such a symmetry is not present in QCD, and thus we should restrict to states invariant under this SO(5) in order to extract QCD physics. This is ensured by only turning on the D8 brane gauge fields which are $SO(5)$ singlets, namely

$$F = \frac{1}{2} F_{\mu\nu}\, \mathrm{d}\, x^\mu \wedge \mathrm{d}\, x^\nu + F_{\mu z}\, \mathrm{d}\, x^\mu \wedge \mathrm{d}\, z , \quad \mu, \nu \in \{0, 1, 2, 3\} , \tag{A.8}$$

i.e. F has no components along S^4 directions. The holographic interpretation of the background $U(N_f)_L \times U(N_f)_R$ gauge fields is given by the asymptotic value of the D8 brane

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worldvolume gauge field along the z direction

$$A_{L\mu}(x^\mu) = \lim_{z\to\infty} A_\mu(x^\mu, z)\,, \quad A_{R\mu}(x^\mu) = \lim_{z\to-\infty} A_\mu(x^\mu, z)\,. \tag{A.9}$$

Writing down a 5d anomaly theory, requires an extension of $A_{L,R}$ to 5d connections. The natural candidate for such an extension is

$$A_\mu(x^\rho, z) = \begin{cases} A^L_\mu(x^\rho, z) & z \geq 0 \\ A^R_\mu(x^\rho, -z) & z \leq 0 \end{cases}, \quad A_z(x^\mu, z) = \begin{cases} A^L_z(x^\rho, z) & z \geq 0 \\ -A^R_z(x^\rho, -z) & z \leq 0 \end{cases} \tag{A.10}$$

In addition to the SO(5) isometry already discussed, the background (A.2) contains further symmetries which are not physical, namely $U(1)_\tau$ isometry generated by shifts of the τ coordinate, and a $\mathbb{Z}_2^{(\tau)}$ generated by inversion of the τ coordinate, dubbed τ-parity.

In the presence of D8 branes, $U(1)_\tau$ is explicitly broken by the D8 brane embedding (A.7). But since the glueball sector is unaffected by the presence of the massless quarks at least in the leading order in the 't Hooft limit, one might hope to impose invariance under $U(1)_\tau$. This is challenging, if not impossible, because the gauge symmetry on the D8 brane mixes the closed string fields with the open string fields on the D8 brane, as can be seen for instance from (3.32). In the following, we use a smeared description of the D8 brane embedding, which allows us to impose invariance under $U(1)_\tau$.

A.2 Inflow action in the (y, z) coordinates

Written as an integral over the z-direction, the inflow action reads

$$2\pi N \int\limits_{z\in(-\infty,\infty)} \mathrm{CS}_5(A - b_1) = 2\pi N \left(\int\limits_{z\in[0,\infty)} \mathrm{CS}_5(A_L - b_1) - \int\limits_{z\in[0,\infty)} \mathrm{CS}_5(A_R - b_1) \right) \tag{A.11}$$

It is useful to consider the compactification of this inflow action, by gluing the cigar with its orientation reversal, and compactifying $\mathbb{R}^{1,3}$ to a compact M_4. Assuming the existence of a 6-manifold N_6 with boundary $\partial N_6 = M_4 \times S^1$, the inflow action may be formulated as an integral over N_6

$$2\pi N \int_{N_6} \frac{1}{3!} \operatorname{Tr}(F - b_2)^3 = 2\pi N \int_{N_6} \left(\frac{1}{3!} \operatorname{Tr}(F^3) - \frac{1}{2} b_2 \operatorname{Tr}(F^2) + \frac{1}{2} {b_2}^2 \operatorname{Tr} F + \frac{N_f}{3!} {b_2}^3 \right). \tag{A.12}$$

To extract a 5d action on M_5, one uses the expansion on the right hand side, together with the flatness of the Kalb-Ramond fluctuations, $\mathrm{d}\, b_2 = 0$, to arrive at

$$\int_{M_4\times S^1} \left(\mathrm{CS}_5(A) - b_2\, \mathrm{CS}_3(A) + \frac{1}{2} {b_2}^2 \operatorname{Tr}(A) + \frac{N_f}{3!} b_1 {b_2}^2 \right). \tag{A.13}$$

In section 4.2, the gauge field is restricted to the following configuration

$$A = A_\chi \mathbb{1}_{N_f} + \tilde{A}\,, \tag{A.14}$$

with A_χ only turned on along the left-handed component, while $\tilde{A}$ is vector-like. The inflow action for this field configuration on N_6 reads

$$2\pi N \int_{N_6} \frac{1}{3!} \left(\operatorname{Tr}\left[(\tilde{F} - b_2)^3 \right] + 3\, \mathrm{d}\, A_\chi \operatorname{Tr}\left[(\tilde{F} - b_2)^2 \right] + 3\, (\mathrm{d}\, A_\chi)^2 \operatorname{Tr}\left(\tilde{F} - b_2 \right) + (\mathrm{d}\, A_\chi)^3 \right). \tag{A.15}$$

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From here, one can extract a 5d action, by restricting to the boundary to arrive at (4.30). The reader is reminded that the $\mathcal{O}({b_2}^2)$ term in this action, is to be combined with a term linear in u_1, as discussed in (4.19), and subsequent discussions.

A.3 Symmetry restrictions for the fluctuation ansatz

As was just argued, in order to extract physical observables of QCD_4, rather than some artefact carried by the KK tower, one has to project onto the singlet sector of the unwanted symmetries. This is achieved for the $SO(5)$ symmetry, due to the fluctuation ansatz in (4.3), as well as the ansatz for the D8-brane gauge field (A.8). Similarly, projection onto $U(1)_\tau$ and $\mathbb{Z}_2^{(\tau)}$ singlet components of all gauge fields must be imposed. Projection onto $U(1)_\tau$ and $\mathbb{Z}_2^{(\tau)}$ singlet states will play a crucial role in extracting the correct anomaly.

With this in mind, let us postulate that the closed-string sector fluctuations are independent of the τ coordinate. For instance, the B-field fluctuations read

$$B_{\mu\nu} = b_{\mu\nu}(x^\rho, r)\,, \quad B_{\mu r} = b_{\mu r}(x^\rho, r)\,, \quad B_{\mu\tau} = 0\,, \tag{A.16}$$

where following the same conventions as the main text, fluctuations are denoted by small letters. The vanishing of the fluctuations of the $B_{\mu\tau}$ components follows from imposing invariance under $\mathbb{Z}_2^{(\tau)}$. In the absence of D8 branes, one must impose invariance under τ-parity and τ-shift of the remaining supergravity fields Φ, and V. This is required for instance, in order to extract the correct glueball spectrum of Yang-Mills theory [31].

In the presence of D8 brane, one can impose $U(1)_\tau$, and $\mathbb{Z}_2^{(\tau)}$ invariance by smearing the D8 branes. The idea behind smearing [39, 40] is to replace the δ-function appearing in the D8 brane action by a smooth charge distribution. In the specific D4-D8 model considered here, the smeared D8 description was discussed in [41, 42]. Our proposal for the smeared topological action is to replace $\tilde{\Omega}$ in (3.25) with the following smeared version

$$\tilde{\Omega}^{(\text{s})} = -\frac{1}{2\pi}\left(\sum_{n\geq 0}(-1)^n\,\text{CS}_{2n+1}(A_R - b_1) - \sum_{n\geq 0}(-1)^n\,\text{CS}_{2n+1}(A_L - b_1)\right)\text{d}\,\tau \tag{A.17}$$

where the superscript on the left hand side is a reminder that this is a smeared quantity.

It is particularly relevant to our main analysis to consider the consequence of this symmetry ansatz on the fluctuations of Φ_2. First of all, requiring the fluctuation to be invariant under $U(1)_\tau$ implies that the fluctuations must be independent of the τ-coordinate

$$\delta\Phi = \phi(x^\mu, r)\,. \tag{A.18}$$

Next, one can further constrain the fluctuations by considering their transformation under $\mathbb{Z}_2^{(\tau)}$. As discussed in [43], the stringy origin of τ-parity is the combination

$$P_\tau = (-1)^{F_L} I_{y,9}\,, \tag{A.19}$$

where $(-1)^{F_L}$ acts on RR fields as multiplication by -1, and $I_{y,9}$ is an inversion in the x^9 and y coordinates. In particular this means that the components of ϕ_2 have the following transformation

$$P_\tau : \begin{cases} \phi_{MN} \to -\phi_{MN} \\ \phi_{M\tau} \to \phi_{M\tau} \end{cases}\,, \tag{A.20}$$

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where $x^M = (x^\mu, r)$. Therefore, the only allowed non-vanishing components of the fluctuations are

$$\Phi_2 = \phi_{M\tau}\, \mathrm{d}\, x^M \wedge \mathrm{d}\, \tau \tag{A.21}$$

After integrating out the Lagrange multiplier $\hat{v}_3$ in (4.6), and smearing the D8 brane, one may write

$$\phi_2^{(\mathrm{s})} = \mathrm{d}\, u_1 + \frac{1}{2\pi} \mathrm{Tr}(A_L - A_R) \wedge \mathrm{d}\, \tau \tag{A.22}$$

Note that the b-dependence of $\phi_2^{(\mathrm{s})}$ has dropped out and only the axial component of the D8-brane gauge field appears in this expression — this will play a crucial role in our main analysis in section 4. The discussion for Φ_4 is completely analogous. Note that the background value of $\Phi_4 = N\omega_4$ is consistent with τ-parity invariance since I_{y9} flips the orientation on S^4, and Φ_4 is odd under $(-1)^{F_L}$. For the fluctuations, ϕ_{MNPQ} is odd under τ-parity, while $\phi_{MNP\tau}$ is even. Hence, only the latter is allowed to be non-vanishing.

B Standard and democratic formulation of type IIA supergravity

In this section we review the standard and democratic formulations for type IIA supergravity. The two formulations lead to the same equations of motion, though each has certain advantages, depending on the precise application one has in mind. In particular, the democratic formulation makes gauge invariance in the presence of D-branes manifest, while it is not appropriate for studying global or topological properties, since everything is expressed in terms of gauge invariant field strengths. Moreover the duality constraint needs to be imposed after deriving the equations of motion, and so the functional is to be viewed as a pseudo-action. The standard (non-democratic) formulation, can be used to study global properties, as there are topological couplings. However, we are not aware of any successful attempt to write down a fully gauge invariant action that couples the bulk to D-branes using the non-democratic formulation.

B.1 Standard formulation

In the standard textbook formulation of supergravity, the flux sector is governed by the action

$$\begin{aligned} S = \pi \int \sum_{n=0}^{2} G_{2n} \wedge \star G_{2n} - \pi \int H \wedge \star H \\ - \pi \int \left(B G_4^2 - B^2 G_2 G_4 + \frac{B^3}{3}\left(G_0 G_4 + G_2^2\right) - \frac{B^4}{4} G_0 G_2 + \frac{B^5}{20} G_0^2 \right) , \end{aligned} \tag{B.1}$$

where the dilaton is set to zero, the metric is assumed to be flat,[12] and we work in units where $2\pi l_s = 1$. In this action, the field strengths are defined in terms of the potentials C_1, C_3, and B

$$H = \mathrm{d}\, B\,, \quad G_4 = \mathrm{d}\, C_3 - H C_1 + \frac{B^2}{2} G_0\,, \quad G_2 = \mathrm{d}\, C_1 + B G_0\,, \tag{B.2}$$

[12] Our convention for the Hodge star reads

$$\star \omega_p = \frac{(-1)^p}{p!(10-p)!} \epsilon_{\mu_1 \cdots \mu_{10}} \omega^{\mu_1 \cdots \mu_p}\, \mathrm{d}\, x^{\mu_{p+1}} \wedge \cdots \wedge \mathrm{d}\, x^{\mu_{10}}$$

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The following Bianchi identities follow from the definition of the field strengths

$$dH = 0\,, \qquad \mathrm{d}\, G_0 = 0\,, \qquad \mathrm{d}\, G_2 = H G_0\,, \qquad dG_4 = H G_2\,. \tag{B.3}$$

The action (B.1) is invariant under the following gauge transformations

$$C_1 \to C_1 + d\Lambda_0 - G_0\zeta_1\,, \quad C_3 \to C_3 + \mathrm{d}\,\Lambda_2 - \Lambda_0 H_3 - G_0\zeta_1\left(B + \frac{\mathrm{d}\,\zeta}{2}\right), \quad B_2 \to B_2 + d\zeta\,. \tag{B.4}$$

The equations of motion follow from variation of the action (B.1) with respect to the potentials C and B

$$\begin{aligned} d\star G_2 &= -H \wedge \star G_4 \\ d\star G_4 &= H \wedge G_4 \\ d\star H &= -\frac{1}{2} G_4 \wedge G_4 + G_2 \wedge \star G_4 + G_0 \wedge \star G_2 \end{aligned} \tag{B.5}$$

Finally, we recall that the integer quantised fluxes are the so-called Page charges [32]

$$\int_{\Sigma_3} H \in \mathbb{Z}\,, \quad \int_{\Sigma_4}\left(G_4 - BG_2 + \frac{B^2}{2}G_0\right) \in \mathbb{Z}\,, \quad \int_{\Sigma_2}(G_2 - BG_0) \in \mathbb{Z}\,, \quad G_0 \in \mathbb{Z}\,, \tag{B.6}$$

where Σ_p denotes a closed p-dimensional manifold.

B.2 Democratic formulation

In the democratic formulation, the flux sector is comprised only of kinetic terms

$$S = \frac{\pi}{2}\int \mathcal{G} \wedge \star\mathcal{G} - \pi \int H \wedge \star H\,, \tag{B.7}$$

where we have introduced

$$\mathcal{G} = \sum_{n=0}^{5} G_{2n}\,, \quad \mathcal{G} = e^B\left(\mathrm{d}\left(Ce^{-B}\right) + G_0\right), \quad C = \sum_{n=0}^{4} C_{2n+1} \quad H = \mathrm{d}\, B\,, \tag{B.8}$$

and the integral is understood to pick up only the 10-form part of the integrand. From these definitions, the Bianchi identity reads

$$\mathrm{d}\,\mathcal{G} = H \wedge \mathcal{G}\,, \qquad \mathrm{d}\, H = 0\,. \tag{B.9}$$

The equations of motion are obtained by varying (B.7) with respect to C and B respectively

$$\mathrm{d}\star\mathcal{G} = -H \star \mathcal{G}\,, \quad \mathrm{d}\star H = \frac{1}{2}\mathcal{G} \wedge \star\mathcal{G} \tag{B.10}$$

The action (B.7) is invariant under the following gauge symmetries

$$B \to B + \mathrm{d}\,\zeta\,, \qquad C \to C - G_0\zeta\, \mathrm{e}^B\, \varphi(\mathrm{d}\,\zeta) + e^B\, \mathrm{d}\left(\Lambda e^{-B}\right); \quad \Lambda = \sum_{n=0}^{4}\Lambda_{2n}\,, \tag{B.11}$$

where we introduced the function [44]

$$\varphi(x) = \frac{e^x - 1}{x} = \sum_{n\geq 0}\frac{x^n}{(n+1)!} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots\,. \tag{B.12}$$

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In addition to the equations of motion, one must impose the duality constraints

$$\mathcal{G} = \epsilon_0 \star \mathcal{G} \tag{B.13}$$

after deriving the equations of motion. In the above ϵ_0 is the sign operator defined via

$$\epsilon_i A_n = \begin{cases} -A_n & n = i \bmod 4 \\ +A_n & \text{Otherwise} \end{cases} \tag{B.14}$$

where A_n is any n-form. The following useful identities follow easily from the above definition, firstly note that the index of the sign operator is well defined mod 4

$$\epsilon_i = \epsilon_{i\pm 4}\,. \tag{B.15}$$

Given any m-form B_m, we have

$$\epsilon_i \left(A_n \wedge B_m\right) = A_n \wedge \epsilon_{i-n} B_m = \epsilon_{i-m} A_n \wedge B_m \tag{B.16}$$

and

$$\epsilon_i \,\mathrm{d}\, A_n = \mathrm{d}\left(\epsilon_{i-1} A_n\right) \tag{B.17}$$

Flux quantisation. The fluxes are subject to Dirac quantisation conditions which read

$$\int H \in \mathbb{Z}\,, \quad \int e^{-B}\mathcal{G} \in \mathbb{Z} \tag{B.18}$$

B.3 Inclusion of D8 branes

In the presence of D-branes, the action needs to be modified to include the bulk-brane couplings. One elegant way to determine these couplings is to use the constraints from cancellation of chiral anomaly on the worldvolume of certain D-brane intersections [28, 29, 45]. One advantage of the democratic formulation is that it allows for including the coupling to D-branes. Consider, for instance, the coupling to a D8 brane

$$S_{\mathrm{D8}} = S_{\mathrm{DBI}} + S_{\mathrm{CS}}\,. \tag{B.19}$$

The first term is the usual DBI action, which will not play a crucial role in our analysis. The second term is the CS coupling, which can be determined using anomaly inflow arguments [28, 29]. Denoting the coordinate transverse to the D8 brane by y, it is useful to define

$$Y = \mathrm{Tr}\exp\left(F\right)\delta(y)\,\mathrm{d}\, y\,, \tag{B.20}$$

where $F = \mathrm{d}\, A + \mathrm{i}\, A^2$ is the field strength of the D8 brane gauge field. Restricting to field configurations where $H = 0$, one may locally write

$$B_2 = \mathrm{d}\, B_1\,. \tag{B.21}$$

It is then possible to write a local expression

$$Y = -\epsilon_1 \,\mathrm{d}\,\Omega\,, \tag{B.22}$$

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with Ω defined in (3.25). Under the gauge symmetry $A \to gAg^{-1} - \mathrm{i}\, g\, \mathrm{d}\, g^{-1}$ on the D8 brane, Ω transforms as a total derivative

$$\delta\Omega = \mathrm{d}\,\alpha(g, A, B_1)\,. \tag{B.23}$$

Crucially, the bulk RR fields have a non-trivial transformation under this symmetry [28, 29, 45]

$$\delta\left(Ce^{-B}\right) = \alpha(g, A, B_1);\,. \tag{B.24}$$

This is the requirement for cancellation of the chiral anomaly [28, 29]. Accordingly, the kinetic terms need to be modified in order to preserve gauge invariance. This is achieved by replacing

$$\mathcal{G} \to \tilde{\mathcal{G}} = e^B\left(\mathrm{d}\left(Ce^{-B}\right) - \Omega + G_0\right)\,. \tag{B.25}$$

These fluxes satisfy a modified Bianchi identity

$$\mathrm{d}\left(\mathrm{e}^{-B}\,\tilde{\mathcal{G}}\right) = -\,\mathrm{d}\,\Omega \tag{B.26}$$

Invariance under the Kalb-Ramond gauge symmetry $B \to B + \mathrm{d}\,\zeta$ is ensured by postulating that the RR potentials transform as

$$Ce^{-B} \to Ce^{-(B+\mathrm{d}\,\zeta)} - G_0\zeta\varphi(-\,\mathrm{d}\,\zeta)\,. \tag{B.27}$$

The action for the bulk-brane system then takes the following form

$$S = \frac{\pi}{2}\int_{M_{10}} \tilde{\mathcal{G}}\wedge\star\tilde{\mathcal{G}} - \pi\int_{M_{10}} H\wedge\star H + \pi\int_{M_{11}} e^{-B}\mathcal{G}\wedge Y + S_{\mathrm{DBI}}\,, \tag{B.28}$$

where we have written the D8 brane CS term as an integral over an 11d manifold M_{11} that bounds M_{10}, namely $\partial M_{11} = M_{10}$. Gauge invariance of the kinetic terms is manifest, since $\tilde{\mathcal{G}}$ is invariant by construction. There are two equivalent 10d expressions for the D8 brane CS coupling, which differ by total derivatives

$$S^{\mathrm{D8}}_{\mathrm{CS}} = \pi\int_{M_{10}}\left(Ce^{-B}\wedge Y + G_0\Omega_{10}\right) = \pi\int_{M_{10}}\epsilon_{03}\left(e^B\Omega\right)\wedge\tilde{\mathcal{G}} \tag{B.29}$$

where $\epsilon_{03} = \epsilon_0\epsilon_3$. The gauge variation with respect to the D8 brane gauge field is engineered to cancel the fermion anomaly [28, 29].

C Proof of bracket identities (3.17), and (3.18)

This appendix contains detailed derivations of the bracket identities (3.17), and (3.18) stated in the main text. The first identity (3.17) follows from inserting (3.16) in the middle expression in (3.15), so that the left hand side of (3.17) becomes

$$\begin{aligned}\langle\Phi,\Psi\rangle_B &= \int\epsilon_0\left[\mathrm{e}^B\,\Phi - \left(\mathrm{e}^B\,\Phi\right)_>\right]\wedge\mathrm{e}^B\,\Psi \\ &= \int\epsilon_0\,(\Phi)\wedge\Psi - \int\epsilon_0\left(\mathrm{e}^B\,\Phi\right)_>\wedge\left(\mathrm{e}^B\,\Psi\right)_<\end{aligned}\,, \tag{C.1}$$

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where we used the fact that $\epsilon_0\left(\mathrm{e}^{B}\,\Phi\right)=\mathrm{e}^{-B}\,\epsilon_0\left(\Phi\right)$ to arrive at the first term in the second line. At this point, one notes that the first term in the second line of (C.1), does not contribute to the integral, as it does not contain a 10-form component. The remaining term in the second line of (C.1) can be rearranged using the definition of the ϵ_0 operator, as the reader can readily confirm by expanding in components, arriving at

$$\langle\Phi,\Psi\rangle_B=-\int\epsilon_2\left(\mathrm{e}^{B}\,\Psi\right)_<\wedge\left(\mathrm{e}^{B}\,\Phi\right)_>=+\int\epsilon_0\left(\mathrm{e}^{B}\,\Psi\right)_<\wedge\left(\mathrm{e}^{B}\,\Phi\right)_>=\langle\Psi,\Phi\rangle_B\,. \tag{C.2}$$

This establishes a proof of the identity (3.17).

To prove (3.18) one inserts the definition of the bracket (3.12) into the left hand side of (3.18)

$$\begin{aligned}\left\langle\left(\mathrm{e}^{-\hat{B}}\,\Phi\right)_<,\left(\mathrm{e}^{-\hat{B}}\,\Psi\right)_<\right\rangle_{B+\hat{B}}&=\int\epsilon_0\left(\mathrm{e}^{B+\hat{B}}\left(\mathrm{e}^{-\hat{B}}\,\Phi\right)_<\right)_<\wedge\left(\mathrm{e}^{B+\hat{B}}\left(\mathrm{e}^{-\hat{B}}\,\Psi\right)_<\right)_>\\&=\int\epsilon_0\left(\mathrm{e}^{B}\,\Phi\right)_<\wedge\left(\mathrm{e}^{B+\hat{B}}\left(\mathrm{e}^{-\hat{B}}\,\Psi\right)_<\right)\,.\end{aligned} \tag{C.3}$$

In the second line we used the fact that the integral only picks up the 10-form component of the integrand to drop the subscript $>$, and used the identity

$$\left(\mathrm{e}^{B+\hat{B}}\left(\mathrm{e}^{-\hat{B}}\,\Phi\right)_<\right)_<=\left(\mathrm{e}^{B}\,\Phi\right)_<\,. \tag{C.4}$$

One should be careful, that a similar simplification cannot occur in the expression containing the Ψ field, that is

$$\left(\mathrm{e}^{B+\hat{B}}\left(\mathrm{e}^{-\hat{B}}\,\Psi\right)_<\right)_>\neq\left(\mathrm{e}^{B}\,\Psi\right)_>\,, \tag{C.5}$$

because the factor $\mathrm{e}^{\hat{B}}$ contains forms up to degree 10, while $\mathrm{e}^{-\hat{B}}$ is truncated as instructed by the subscript $<$. To proceed further, one uses the analogue of (3.16) for Ψ into the second line of (C.3)

$$\begin{aligned}\left\langle\left(\mathrm{e}^{-\hat{B}}\,\Phi\right)_<,\left(\mathrm{e}^{-\hat{B}}\,\Psi\right)_<\right\rangle_{B+\hat{B}}&=\int\epsilon_0\left(\mathrm{e}^{B}\,\Phi\right)_<\wedge\mathrm{e}^{B+\hat{B}}\left[\mathrm{e}^{-\hat{B}}\,\Psi-\left(\mathrm{e}^{-\hat{B}}\,\Psi\right)_>\right]\\&=\int\epsilon_0\left(\mathrm{e}^{B}\,\Phi\right)_<\wedge\left(\mathrm{e}^{B}\,\Psi\right)-\int\epsilon_0\left(\mathrm{e}^{B}\,\Phi\right)\mathrm{e}^{B+\hat{B}}\left(\mathrm{e}^{-\hat{B}}\,\Psi\right)_>\,,\end{aligned} \tag{C.6}$$

where once again a subscript $<$ is dropped in the second term in the second line, exploiting the fact that the integral only picks up the 10-form component of the integrand. Using the identity, $\epsilon_0\left(\mathrm{e}^{B}\,\Phi\right)\mathrm{e}^{B+\hat{B}}=\epsilon_0\left(\mathrm{e}^{-B-\hat{B}}\,\mathrm{e}^{B}\,\Phi\right)=\epsilon_0\left(\mathrm{e}^{-\hat{B}}\,\Phi\right)$, one finds that the right hand side of (C.6) is equivalent to the right hand side of (3.18).

Data Availability Statement. This article has no associated data or the data will not be deposited.

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